

# Quasi-Hadamard differentiability of general risk functionals and its application

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## Abstract

We apply a suitable modification of the functional delta method to statistical functionals that arise from law-invariant coherent risk measures. To this end we establish differentiability of the statistical functional in a relaxed Hadamard sense, namely with respect to a suitably chosen norm and in the directions of a specifically chosen “tangent space”. We show that this notion of quasi-Hadamard differentiability yields both strong laws and limit theorems for the asymptotic distribution of the plug-in estimators. Our results can be regarded as a contribution to the statistics and numerics of risk measurement and as a case study for possible refinements of the functional delta method through fine-tuning the underlying notion of differentiability

**Keywords:** Functional delta method; quasi-Hadamard derivative; law-invariant coherent risk measure; Kusuoka representation; weak limit theorem; strong limit theorem

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# 1. Introduction

Let  $X$  be a random variable describing the future profits and losses of a financial position. When assessing its risk,  $\rho(X)$ , in terms of a risk measure  $\rho$  it is common to estimate  $\rho(X)$  by means of a Monte Carlo procedure or from a sequence of historical data. This problem is well-posed when  $\rho$  is “law-invariant” in the sense that there exists a functional  $\mathcal{R}_\rho$  such that  $\rho(X) = \mathcal{R}_\rho(F_X)$  for  $F_X$  denoting the distribution function of  $X$ . In this case, a natural estimate for  $\rho(X)$  is given by  $\mathcal{R}_\rho(\hat{F}_n)$ , where  $\hat{F}_n$  is the empirical distribution function of the given data or another suitable estimate for  $F_X$ . In recent years, the statistical properties of the plug-in estimator  $\mathcal{R}_\rho(\hat{F}_n)$  and of the statistical functional  $\mathcal{R}_\rho(\cdot)$  have been the subject of a number of studies. This includes studies on consistency and robustness [10, 22], elicibility [18, 42], and weak limit theorems [6, 8, 27]. Here we continue and extend the latter class of studies by suitably adapting the modified functional delta method from [8] to the case of a general law-invariant coherent risk measure  $\rho$ .

The functional delta method allows to lift a functional weak limit theorem from the level of the input data to the level of the plug-in estimators; see [36, 29, 14, 17] for early references. This means in our present context and under suitable technical assumptions that if there are numbers  $(r_n)_{n \in \mathbb{N}}$  such that  $r_n(\hat{F}_n - F_X)$  converges in law to some random variable  $B^\circ$ , then  $r_n(\mathcal{R}_\rho(\hat{F}_n) - \mathcal{R}_\rho(F_X))$  converges in law to the random variable  $\dot{\mathcal{R}}_{\rho, F_X}(B^\circ)$ , where  $\dot{\mathcal{R}}_{\rho, F_X}(\cdot)$  is a suitable derivative of the statistical functional  $\mathcal{R}_\rho$  at  $F_X$ . The problem of establishing a weak limit theorem for the plug-in estimators  $\mathcal{R}_\rho(\hat{F}_n)$  is thus reduced to the following two independent steps:

- (a) showing a weak limit theorem for the sequence  $(\hat{F}_n)$ ;
- (b) proving that  $\mathcal{R}_\rho$  admits the derivative  $\dot{\mathcal{R}}_{\rho, F_X}(\cdot)$ .

When carrying out this program it is crucial to fine-tune the notion of differentiability of the functional  $\mathcal{R}_\rho$ . On the one hand, this notion needs to be sufficiently strong so that in combination with step (a) it yields the desired weak limit theorem for  $(\mathcal{R}_\rho(\hat{F}_n))_{n \in \mathbb{N}}$ . On the other hand, the derivative  $\dot{\mathcal{R}}_{\rho, F_X}(\cdot)$  should exist for a wide class of risk measures  $\rho$  and distribution functions  $F_X$ , which will only be the case when differentiability is understood in a sufficiently weak sense. The classical notion used in the literature is Hadamard differentiability, which is stronger than differentiability in the sense of Gâteaux but weaker than Fréchet differentiability (cf. e.g. [34, 38]). But, as observed in [8], this notion is still too strong to be applied to some of the most common law-invariant coherent risk measures  $\rho$ . This includes in particular many distortion risk measures  $\rho_g$  that are defined as follows for concave, nondecreasing distortion functions  $g : [0, 1] \rightarrow [0, 1]$  satisfying  $g(0) = 0$  and  $g(1) = 1$ ,

$$\rho_g(X) = \mathcal{R}_{\rho_g}(F_X) = \int_{-\infty}^0 g(F_X(x)) dx - \int_0^\infty (1 - g(F_X(x))) dx. \quad (1)$$

Therefore, a relaxed notion of *quasi-Hadamard differentiability* was proposed in [8]. This notion differs from classical Hadamard differentiability mainly by taking derivatives only in the directions of a relatively small “tangent space” equipped with a suitably weighted norm and by relying on convergence with respect to this weighted norm. In contrast to the existing literature on tangential Hadamard differentiability in the context of the functional delta method, no single distribution function will have a finite length w.r.t. the imposed (weighted) norm, meaning that no single distribution function will belong to the “tangent space”.

In this paper, our goal is to extend the functional delta method based on quasi-Hadamard differentiability to a general class of law-invariant coherent risk measures. It is known that any

such risk measure  $\rho$  admits the following Kusuoka representation,

$$\rho(X) = \sup_{g \in \mathcal{G}} \rho_g(X), \quad (2)$$

where  $\mathcal{G}$  is a class of distortion functions and  $\rho_g$  denotes the distortion risk measure (1) for  $g \in \mathcal{G}$ . The so-called expectiles risk measures are examples for risk measures that are of the form (2) but not of the form (1); see Example 2.10. The Kusuoka representation (2) will also play an important role in our formulas for the quasi-Hadamard derivatives of the statistical functionals  $\mathcal{R}_\rho$ .

By analyzing quasi-Hadamard differentiability of general law-invariant coherent risk measures we are pursuing two different objectives. On the one hand we are aiming to contribute to the asymptotic analysis of plug-in estimators in risk measurement, with a view toward statistical inference, Monte Carlo computation, and optimization. On the other hand, we wish to provide a case study for possible refinements of the functional delta method through fine-tuning the underlying notion of differentiability. In this latter respect, the scope of our analysis is not limited to applications in risk assessment.

Our main results on the quasi-Hadamard differentiability of statistical functionals  $\mathcal{R}_\rho$  will be given in Theorems 2.4 and 2.7 under two different sets of assumptions. These assumptions will be illustrated in Sections 2.1 and 2.2 by means of a number of examples. Our applications to statistical inference are given in Section 3. Specifically, our weak limit theorem for the sequence of plug-in estimator will be stated in Section 3.1 along with discussions for the cases of independent, weakly dependent, or strongly dependent data. In Section 3.2 a strong law of the form  $r_n(\mathcal{R}_\rho(\hat{F}_n) - \mathcal{R}_\rho(F_0)) \rightarrow 0$  will be stated and discussed. The proofs of our main results are contained in Section 4. In Appendix A we recall the notion of quasi-Hadamard differentiability and state some auxiliary results in a general setting. Appendix B contains an auxiliary result on Skorohod spaces.

## 2. Main results

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be an atomless probability space and denote by  $L^p(\Omega, \mathcal{F}, \mathbb{P})$  the usual  $L^p$ -spaces for  $p \in [1, \infty]$ . Let  $\mathcal{X}$  be a subspace of  $L^1(\Omega, \mathcal{F}, \mathbb{P})$  containing  $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ . An element  $X$  of  $\mathcal{X}$  will be interpreted as the P&L of a financial position. As usual (cf. e.g. [3, 16]), we will say that a map  $\rho : \mathcal{X} \rightarrow \mathbb{R}$  is a *coherent risk measure* if it is

- monotone:  $\rho(X) \geq \rho(Y)$  for all  $X, Y \in \mathcal{X}$  with  $X \leq Y$ ,
- cash-invariant:  $\rho(X + m) = \rho(X) - m$  for all  $X \in \mathcal{X}$  and  $m \in \mathbb{R}$ ,
- subadditive:  $\rho(X + Y) \leq \rho(X) + \rho(Y)$  for all  $X, Y \in \mathcal{X}$ ,
- positively homogenous:  $\rho(\lambda X) = \lambda \rho(X)$  for all  $X \in \mathcal{X}$  and  $\lambda \geq 0$ .

A coherent risk measure  $\rho$  will be called *law-invariant* if  $\rho(X) = \rho(Y)$  whenever  $X$  and  $Y$  have the same law under  $\mathbb{P}$ .

To give a typical example, let  $g : [0, 1] \rightarrow [0, 1]$  be a concave distortion function, i.e. a concave and nondecreasing function with  $g(0) = 0$  and  $g(1) = 1$ . The *distortion risk measure* associated with  $g$  is defined by

$$\rho_g(X) := \int_{-\infty}^0 g(F_X(x)) dx - \int_0^\infty (1 - g(F_X(x))) dx \quad (3)$$

for every random variable  $X \in L^0(\Omega, \mathcal{F}, \mathbb{P})$  satisfying  $\int_0^\infty g(1 - F_{|X|}(x)) dx < \infty$ , where  $F_X$  and  $F_{|X|}$  denote the distribution functions of  $X$  and  $|X|$ , respectively. The set  $\mathcal{X}$  of all such random variables forms a linear subspace of  $L^1(\Omega, \mathcal{F}, \mathbb{P})$ ; this follows from [12, Proposition 9.5] and [16, Proposition 4.75]. It is known that  $\rho_g$  is a law-invariant coherent risk measure; see, for instance, [39]. If specifically  $g(t) = (t/\alpha) \wedge 1$  for any fixed  $\alpha \in (0, 1)$ , then we have  $\mathcal{X} = L^1(\Omega, \mathcal{F}, \mathbb{P})$  and  $\rho_g$  is nothing but the Average Value at Risk at level  $\alpha$ . The latter is defined by

$$\text{AV@R}_\alpha(X) := \frac{1}{\alpha} \int_0^\alpha \text{V@R}_s(X) ds, \quad X \in L^1(\Omega, \mathcal{F}, \mathbb{P}), \quad (4)$$

where  $\text{V@R}_s(X) := -F_X^\rightarrow(s)$  is the Value at Risk at level  $s$ . Here and elsewhere  $F^\leftarrow(s) := \inf\{x \in \mathbb{R} : F(x) \geq s\}$  and  $F^\rightarrow(s) := \inf\{x \in \mathbb{R} : F(x) > s\}$  denote the left-continuous and the right-continuous inverses of  $F$  at  $s$ , respectively.

For any coherent risk measure  $\rho : \mathcal{X} \rightarrow \mathbb{R}$  we can define a function  $g_\rho : [0, 1] \rightarrow [0, 1]$  by

$$g_\rho(t) := \rho(-B_{1,t}), \quad t \in [0, 1], \quad (5)$$

where  $B_{1,t}$  is a Bernoulli random variable with expectation  $t$ . Clearly,  $g_\rho$  is distortion function, i.e. a nondecreasing function  $g_\rho : [0, 1] \rightarrow [0, 1]$  with  $g_\rho(0) = 0$  and  $g_\rho(1) = 1$ , and we will refer to  $g_\rho$  as the *distortion function associated with  $\rho$* . For an alternative representation of  $g_\rho$  see part (ii) of Theorem 2.2 below.

Distortion risk measures w.r.t. concave distortion functions are known to be building blocks of general law-invariant coherent risk measures on  $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ . This is the so called Kusuoka representation (cf. [24]). More precisely, there exists some set  $\mathcal{G}$  of concave distortion functions such that  $\rho = \sup_{g \in \mathcal{G}} \rho_g$ . Recently, the Kusuoka representation has been extended to law-invariant coherent risk measures on more general spaces  $\mathcal{X}$  under the following technical conditions; cf. [6] or [23].

**Assumption 2.1** *Let  $\mathcal{X}$  be a Stonean vector lattice, i.e.  $X \wedge Y, X \vee Y \in \mathcal{X}$  for all  $X, Y \in \mathcal{X}$ . Moreover, let  $\rho$  be a law-invariant coherent risk measure on  $\mathcal{X}$  and assume that the following conditions hold:*

- (a)  $\lim_{k \rightarrow \infty} \rho(-(X - k)^+) = 0$  for all nonnegative  $X \in \mathcal{X}$ .
- (b)  $\lim_{t \downarrow 0} g_\rho(t) = 0$ .

Remark 2.8 and Examples 2.9–2.11 below will illustrate Assumption 2.1. The next theorem deals with the above-mentioned extension of the Kusuoka representation for law-invariant coherent risk measures on general spaces  $\mathcal{X}$ . Under Assumption 2.1, the representing concave distortion functions satisfy some additional useful properties.

**Theorem 2.2** *Suppose that Assumption 2.1 holds. Then we can find some set  $\mathcal{G}_\rho$  of continuous concave distortion functions such that the following assertions hold:*

- (i)  $\mathcal{G}_\rho$  is compact w.r.t. the uniform metric.
- (ii)  $g_\rho = \sup_{g \in \mathcal{G}_\rho} g$ .
- (iii)  $\sup_{g \in \mathcal{G}_\rho} g'(t) \leq g_\rho(\gamma t)/(\gamma t)$  for all  $\gamma, t \in (0, 1)$ .
- (iv)  $\rho(X) = \sup_{g \in \mathcal{G}_\rho} \rho_g(X)$  for all  $X \in \mathcal{X}$ .

(v) For every  $X \in \mathcal{X}$ , the mapping  $\mathcal{G}_\rho \rightarrow \mathbb{R}$ ,  $g \mapsto \rho_g(X)$  is lower semicontinuous w.r.t. the uniform metric, and it is even continuous if in addition  $\int_{-\infty}^0 g_\rho(\gamma F_X(x)) dx < \infty$  holds for some  $\gamma \in (0, 1)$ .

The proof of Theorem 2.2 can be found in Section 4.1. If  $\rho$  is a distortion risk measure associated with a continuous concave distortion function  $g$ , then, of course,  $g_\rho = g$  and  $\mathcal{G}_\rho$  reduces to the singleton  $\mathcal{G}_\rho = \{g\}$ .

Now, let  $\phi : \mathbb{R} \rightarrow [1, \infty)$  be a *weight function*, i.e. a continuous function that is nondecreasing on  $(-\infty, 0]$  and nonincreasing on  $[0, \infty)$ . Let  $D_\phi$  be the space of all càdlàg functions  $v$  on  $\mathbb{R}$  with  $\|v\|_\phi := \|v\phi\|_\infty < \infty$ , where  $\|f\|_\infty := \sup_{x \in \mathbb{R}} |f(x)|$  denotes the sup-norm of a real-valued function  $f$  on  $\mathbb{R}$ . Let  $\mathbb{F}_\mathcal{X}$  be the set of distribution functions of the random variables from  $\mathcal{X}$ . For any given  $F_0 \in \mathbb{F}_\mathcal{X}$ , we denote by  $D_{\phi, F_0}$  the subspace of all  $v \in D_\phi$  vanishing outside  $[F_0^{\rightarrow}(0), F_0^{\leftarrow}(1)]$ , and by  $C_{\phi, F_0}$  the subspace of all functions in  $D_{\phi, F_0}$  whose discontinuity points on  $(F_0^{\rightarrow}(0), F_0^{\leftarrow}(1))$  are discontinuity points of  $F_0$ . Both  $C_{\phi, F_0}$  and  $D_{\phi, F_0}$  will be equipped with the norm  $\|\cdot\|_\phi$ .

**Assumption 2.3** Let  $\rho$  be a law-invariant coherent risk measure on  $\mathcal{X}$ , and  $g_\rho$  the distortion function associated with  $\rho$  as defined in (5). Let  $F_0 \in \mathbb{F}_\mathcal{X}$ ,  $\phi$  be a weight function as above, and assume that the following conditions holds:

- (a) There exists a finite set  $D(F_0)$  of real numbers such that  $F_0$  is continuously differentiable with strictly positive derivative on  $(F_0^{\rightarrow}(0), F_0^{\leftarrow}(1)) \setminus D(F_0)$ .
- (b)  $\int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} g_\rho(\gamma F_0(x)) / (F_0(x)\phi(x)) dx < \infty$  for some  $\gamma \in (0, 1)$ .

Remarks 2.12 and 2.13 below will illustrate Assumption 2.3(b). When  $\rho$  is law-invariant we may associate with  $\rho$  a statistical functional  $\mathcal{R}_\rho$  defined on the class  $\mathbb{F}_\mathcal{X}$  of all distribution functions of elements of  $\mathcal{X}$  via

$$\mathcal{R}_\rho(F_X) := \rho(X), \quad X \in \mathcal{X}. \quad (6)$$

The following theorem involves the notion of quasi-Hadamard differentiability, which is recalled in Definition A.1 in the Appendix A. In our specific setting, the roles of  $\mathbf{V}$ ,  $\mathbf{V}_0$ ,  $\mathbb{C}_0$  and  $\mathbf{V}'$  from Definition A.1 are played by  $D$ ,  $D_{\phi, F_0}$ ,  $C_{\phi, F_0}$  and  $\mathbb{R}$ , respectively. Here  $D$  is the space of all bounded càdlàg functions. That is, quasi-Hadamard differentiability of  $\mathcal{R}_\rho$  at  $F_0$  tangentially to  $C_{\phi, F_0} \langle D_{\phi, F_0} \rangle$  means that we can find some continuous map  $\dot{\mathcal{R}}_{\rho, F_0} : C_{\phi, F_0} \rightarrow \mathbb{R}$  such that

$$\lim_{n \rightarrow \infty} \left| \frac{\mathcal{R}_\rho(F_0 + h_n v_n) - \mathcal{R}_\rho(F_0)}{h_n} - \dot{\mathcal{R}}_{\rho, F_0}(v) \right| = 0$$

holds for every triplet  $(v, (v_n), (h_n))$  with  $v \in C_{\phi, F_0}$ ,  $(v_n) \subset D_{\phi, F_0}$  satisfying  $\|v_n - v\|_\phi \rightarrow 0$  and  $(F_0 + h_n v_n) \subset \mathbb{F}_\mathcal{X}$ , and  $(h_n) \subset (0, \infty)$  satisfying  $h_n \rightarrow 0$ .

**Theorem 2.4** Suppose that Assumptions 2.1 and 2.3 hold, and let  $\mathcal{G}_\rho$  be given by Theorem 2.2. Then the functional  $\mathcal{R}_\rho$  defined in (6) is quasi-Hadamard differentiable at  $F_0$  tangentially to  $C_{\phi, F_0} \langle D_{\phi, F_0} \rangle$  with quasi-Hadamard derivative  $\dot{\mathcal{R}}_{\rho, F_0}$  given by

$$\dot{\mathcal{R}}_{\rho, F_0}(v) := \lim_{\varepsilon \downarrow 0} \sup_{g \in \mathcal{G}_\rho(F_0, \varepsilon)} \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} g'(F_0(x)) v(x) dx, \quad v \in C_{\phi, F_0}, \quad (7)$$

where  $\mathcal{G}_\rho(F_0, \varepsilon)$  denotes the set of all  $g \in \mathcal{G}_\rho$  satisfying  $\mathcal{R}_\rho(F_0) - \varepsilon \leq \mathcal{R}_g(F_0)$ , and  $g'$  stands for the right-sided derivative of  $g$ .

If in addition  $\int_{-\infty}^0 g_\rho(\delta F_0(x)) dx < \infty$  holds for some  $\delta \in (0, 1)$ , then the set  $\mathcal{G}_\rho(F_0)$  of all  $g \in \mathcal{G}_\rho$  satisfying  $\mathcal{R}_\rho(F_0) = \mathcal{R}_g(F_0)$  is nonempty, and the quasi-Hadamard derivative has the following form

$$\dot{\mathcal{R}}_{\rho, F_0}(v) := \sup_{g \in \mathcal{G}_\rho(F_0)} \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} g'(F_0(x)) v(x) dx, \quad v \in C_{\phi, F_0}. \quad (8)$$

The proof of Theorem 2.4 can be found in Section 4.3.

**Remark 2.5** Under condition (b) from Assumptions 2.3, we obtain  $\int_{-\infty}^0 g_\rho(\delta F_0(x)) dx < \infty$  for some  $\delta \in (0, 1)$  if  $F_0 \phi$  is bounded on  $(-\infty, 0)$ . Indeed, choosing  $\gamma \in (0, 1)$  from part (b) of Assumptions 2.3, we may conclude with Hölder's inequality that the function  $g_\rho(\gamma F_0) \mathbb{1}_{(F_0^{\rightarrow}(0) \wedge 0, 0)} = ((g_\rho(\gamma F_0))/(F_0 \phi)) F_0 \phi \mathbb{1}_{(F^{\rightarrow}(0) \wedge 0, 0)}$  is integrable w.r.t. the Lebesgue measure on  $\mathbb{R}$ .  $\diamond$

For distortion risk measures  $\rho = \rho_g$  with continuous distortion function  $g$ , Theorem 2.4 can be improved and the assumptions on  $\phi$  and  $F_0$  can be relaxed as follows. In this case we clearly have  $g_\rho = g$ .

**Assumption 2.6** Let  $\rho_g : \mathcal{X} \rightarrow \mathbb{R}$  be a distortion risk measure associated with a continuous concave distortion function  $g$  as defined in (3). Moreover, let  $F_0 \in \mathbb{F}_{\mathcal{X}}$ ,  $\phi$  be a weight function, and assume that the following conditions holds:

- (a) The set of points  $x \in (F_0^{\rightarrow}(0), F_0^{\leftarrow}(1))$  for which  $g$  is not differentiable at  $F_0(x)$  has Lebesgue measure zero.
- (b) Assumption 2.3 (b) holds for  $\rho = \rho_g$ .

**Theorem 2.7** Suppose that Assumption 2.6 holds. Then the functional  $\mathcal{R}_g = \mathcal{R}_{\rho_g}$  is quasi-Hadamard differentiable at  $F_0$  tangentially to  $C_{\phi, F_0} \langle D_{\phi, F_0} \rangle$  with quasi-Hadamard derivative  $\dot{\mathcal{R}}_g$  given by

$$\dot{\mathcal{R}}_g(v) := \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} g'(F_0(x)) v(x) dx, \quad v \in C_{\phi, F_0}, \quad (9)$$

where  $g'$  denotes as before the right-sided derivative of  $g$ .

The proof of Theorem 2.7 can be found in Section 4.4. Theorem 2.7 partially generalizes Theorem 2.2 in [8] where it was assumed that  $g'$  is bounded. On the other hand, in [8] the distortion function  $g$  was not required to be concave.

## 2.1. Illustration of Assumption 2.1

**Remark 2.8** It is worth pointing out that conditions (a)–(b) in Assumption 2.1 are always fulfilled if one can find a complete norm  $\|\cdot\|$  on the Stonean vector lattice  $\mathcal{X}$  such that, for every  $X, Y, X_1, X_2, \dots \in \mathcal{X}$ , we have  $\|X\| \leq \|Y\|$  when  $|X| \leq |Y|$ , and  $\lim_{k \rightarrow \infty} \|X_k\| = 0$  when  $X_k \uparrow 0$   $\mathbb{P}$ -a.s. As already discussed in [6, Remark 3.2] this follows from results in [32]. General classes of random variables meeting these requirements are  $L^p(\Omega, \mathcal{F}, \mathbb{P})$  equipped with the usual  $L^p$ -norm for  $p \in [1, \infty]$ , and, more generally, the Orlicz heart  $H^\psi(\Omega, \mathcal{F}, \mathbb{P})$  equipped with the Luxemburg norm associated with a continuous Young function  $\psi$ ; for more details see [6, Remark 3.2].  $\diamond$

The discussion in the preceding Remark 2.8 shows that every law-invariant coherent risk measure  $\rho$  which is defined on some  $L^p(\Omega, \mathcal{F}, \mathbb{P})$  for some  $p \in [1, \infty]$ , or, more generally, on some Orlicz heart  $H^\psi(\Omega, \mathcal{F}, \mathbb{P})$  for some continuous Young function  $\psi$ , is covered by Theorem 2.4. Examples are the one-sided  $p$ th moment risk measure defined on  $L^p(\Omega, \mathcal{F}, \mathbb{P})$ ,  $p \in [1, \infty)$ , the expectiles-based risk measure defined on  $L^2(\Omega, \mathcal{F}, \mathbb{P})$ , and the Haezendonck–Goovaerts risk measure associated with a continuous Young function  $\psi$  defined on  $H^\psi(\Omega, \mathcal{F}, \mathbb{P})$ . For details see the following Examples 2.9–2.11 and [6, Examples 3.2–3.4].

**Example 2.9** Given  $a \in (0, 1]$  and  $p \in [1, \infty)$ , the *one-sided  $p$ th moment risk measure* is defined by

$$\rho(X) := -\mathbb{E}[X] + a\|(X - \mathbb{E}[X])^-\|_p, \quad X \in L^p(\Omega, \mathcal{F}, \mathbb{P}).$$

In [15] it has been shown that  $\rho$  is a law-invariant coherent risk measure. The associated distortion function is given by  $g_\rho(t) = t + a(1-t)t^{1/p}$ ,  $t \in [0, 1]$ . Since  $\rho$  is defined on  $L^p(\Omega, \mathcal{F}, \mathbb{P})$ , it satisfies conditions (a)–(b) in Assumption 2.1; cf. Remark 2.8. But by Lemma A.5 in [23] it is not a distortion risk measure for  $a > 0$ .  $\diamond$

**Example 2.10** In [4] it has been pointed out that expectiles, genuinely introduced in [26], may be viewed as law-invariant coherent risk measures. The *expectiles-based risk measure* associated with  $\alpha \in [1/2, 1)$  is defined by

$$\rho(X) := \operatorname{argmin} \left\{ (1-\alpha)\|((-X) - x)^-\|_2^2 + \alpha\|((-X) - x)^+\|_2^2 : x \in \mathbb{R} \right\}, \quad X \in L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

The associated distortion function is given by  $g_\rho(t) = (\alpha t)/(1 - \alpha + t(2\alpha - 1))$ ,  $t \in [0, 1]$ . Since  $\rho$  is defined on  $L^2(\Omega, \mathcal{F}, \mathbb{P})$ , it satisfies conditions (a)–(b) in Assumption 2.1; cf. Remark 2.8. The set  $\mathcal{G}_\rho$  corresponding to the Kusuoka representation of  $\rho$  is identified in [11, Theorem 8]. This result implies in particular that  $\rho$  is not a distortion risk measure unless  $\alpha = 1/2$ .  $\diamond$

**Example 2.11** Let  $\psi$  be a strictly increasing continuous Young function with  $\psi(1) = 1$ . By Young function we mean a nondecreasing, unbounded, convex function  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\psi(0+) = 0$ . Let  $\mathcal{X} = H^\psi(\Omega, \mathcal{F}, \mathbb{P})$  be the Orlicz heart associated with  $\psi$ ,

$$H^\psi(\Omega, \mathcal{F}, \mathbb{P}) := \{X \in L^0 : \mathbb{E}[\Psi(c|X|)] < \infty \text{ for all } c > 0\},$$

and fix  $\alpha \in (0, 1)$ . It was shown in [19] that for every  $X \in H^\psi(\Omega, \mathcal{F}, \mathbb{P})$  and every  $x \in \mathbb{R}$  with  $\mathbb{P}[X > x] > 0$  there exists a unique real number  $\pi_\alpha^\psi(X, x) > x$  such that

$$\mathbb{E} \left[ \psi \left( \frac{(X - x)^+}{\pi_\alpha^\psi(X, x) - x} \right) \right] = 1 - \alpha.$$

Therefore we may uniquely define

$$\rho(X) := \inf \{ \pi_\alpha^\psi(-X, x) : x \in \mathbb{R} \text{ with } \mathbb{P}[-X > x] > 0 \}, \quad X \in H^\psi(\Omega, \mathcal{F}, \mathbb{P}).$$

It is known from [5, 23] that  $\rho$  is a law-invariant coherent risk measure, sometimes referred to as *Haezendonck–Goovaerts risk measure* associated with  $\psi$  and  $\alpha$ . For the associated distortion function we have

$$g_\rho(t) \leq 1 \wedge \left( t + \frac{1-t}{\psi^{-1}((1-\alpha)/t)} \right) \quad \text{for all } t \in [0, 1], \quad (10)$$

where  $\psi^{-1}$  denotes the inverse function of  $\psi$ . Since  $\rho$  is defined on  $H^\psi(\Omega, \mathcal{F}, \mathbb{P})$ , it satisfies conditions (a)–(b) in Assumption 2.1; cf. Remark 2.8. However, as explained in the Appendix of [23],  $\rho$  is not a distortion risk measure in general.  $\diamond$

## 2.2. Illustration of Assumption 2.3 (b)

**Remark 2.12** If  $-\infty < F_0^{\rightarrow}(0)$  and  $F_0^{\leftarrow}(1) < \infty$ , i.e. if  $dF_0$  has compact support, then the integrability condition (b) in Assumption 2.3 is fulfilled for every weight function  $\phi$ . That is, in this case we may assume without loss of generality that  $\phi = \mathbb{1}$ .  $\diamond$

**Remark 2.13** Assume that  $-\infty = F_0^{\rightarrow}(0)$  and  $F_0^{\leftarrow}(1) = \infty$ . If  $\limsup_{t \rightarrow 0+} g_\rho(t)/t^\beta < \infty$ , which equivalently means that  $g_\rho(t) \leq C t^\beta$ ,  $t \in [0, 1]$ , for some constants  $C \in (0, \infty)$  and  $\beta \in (0, 1]$ , then the integrability condition (b) in Assumption 2.3 is implied by

$$\int_{-\infty}^{\infty} \frac{1}{F_0(x)^{1-\beta} \phi(x)} dx < \infty. \quad (11)$$

Conditions (b) in Assumption 2.3 and the integrability condition (11) are even equivalent if in addition  $\liminf_{t \rightarrow 0+} g_\rho(t)/t^\beta > 0$  holds. In particular, for every weight function  $\phi$ , Assumption 2.3 (b) and (11) are equivalent if  $\lim_{t \rightarrow 0+} g_\rho(t)/t^\beta \in (0, \infty)$ .

(i) For the one-sided  $p$ th moment risk measure defined in Example 2.9, with  $p \in [1, \infty)$ , we have  $\lim_{t \rightarrow 0+} g_\rho(t)/t^{1/p} \in (0, \infty)$ . Thus, for every weight function  $\phi$ , Assumption 2.3 (b) is equivalent to

$$\int_{-\infty}^{\infty} \frac{1}{F_0(x)^{(p-1)/p} \phi(x)} dx < \infty.$$

(ii) For the expectiles-based risk measure as defined in Example 2.10, with  $\alpha \in [1/2, 1)$ , we have  $\lim_{t \rightarrow 0+} g_\rho(t)/t = \alpha/(1-\alpha) \in (0, \infty)$ . Thus, for every weight function  $\phi$ , Assumption 2.3 (b) is equivalent to

$$\int_{-\infty}^{\infty} \frac{1}{\phi(x)} dx < \infty.$$

(iii) For the Haerendonck–Goovaerts risk measure as defined in Example 2.11, with  $\psi$  and  $\alpha \in (0, 1)$ , we may conclude from (10) that  $\limsup_{t \rightarrow 0+} g_\rho(t)/t^\beta < \infty$  for some  $\beta \in (0, 1]$  whenever  $\liminf_{t \rightarrow 0+} \psi^{-1}((1-\alpha)/t)t^\beta > 0$ . The latter condition may be described equivalently by the condition  $\liminf_{t \rightarrow 0+} \psi^{-1}(1/t)t^\beta > 0$ , and is satisfied if  $\limsup_{x \rightarrow \infty} \psi(x)/x^{1/\beta} < \infty$ . Thus, condition (b) in Assumption 2.3 is satisfied if (11) holds with this choice of  $\beta$ .

(iv) For  $\rho = \text{AV@R}_\alpha$  defined in (4) we obviously have  $g_\rho(t) = g(t) = (t/\alpha) \wedge 1$ . So we have, in particular,  $\lim_{t \rightarrow 0+} g_\rho(t)/t = 1/\alpha \in (0, \infty)$ . Thus, Assumption 2.3 (b) is satisfied for every  $F_0 \in \mathbb{F}_{L^1(\Omega, \mathcal{F}, \mathbb{P})}$  if and only if

$$\int_{-\infty}^{\infty} \frac{1}{\phi(x)} dx < \infty.$$

$\diamond$



### 3. Application to statistical inference

The quasi-Hadamard derivative  $\dot{\mathcal{R}}_{\rho, F_0}$  of  $\mathcal{R}_\rho$  evaluated at  $G - F_0$  can be seen as a measure for the sensitivity of a sequence of plug-in estimators for  $\mathcal{R}_\rho(F_0)$  w.r.t. a contamination  $F_{0,h} := (1-h)F_0 + hG$  of  $F_0$  (with  $h$  small) for some given distribution function  $G$ ; cf. the discussion in Section 5 of [21]. It also facilitates the derivation of weak and strong limit theorems for plug-in estimators of  $\mathcal{R}_\rho(F_0)$ , and establishing such limit theorems will be our goal in this section. For any given  $F_0 \in \mathbb{F}_\mathcal{X}$ , we equip  $D_{\phi, F_0}$  with the trace  $\sigma$ -algebra  $\mathcal{D}_{\phi, F_0} := \mathcal{D} \cap D_{\phi, F_0}$ , where  $\mathcal{D}$  denotes the  $\sigma$ -algebra generated by the coordinate projections on the space  $D$  of all bounded càdlàg functions on  $\mathbb{R}$ . Notice that  $\mathcal{D}_{\phi, F_0}$  coincides with the ball  $\sigma$ -algebra on  $(D_{\phi, F_0}, \|\cdot\|_\phi)$ . This fact may be obtained for  $\phi = \mathbb{1}$  following [28, Problem IV.2.2]. For general  $\phi$ , one uses that  $D_{\phi, F_0}$  and  $D_{\mathbb{1}, F_0}$  are isometrically isomorphic for their respective norms. As a consequence, any  $\|\cdot\|_\phi$ -closed and separable subset of  $D_{\phi, F_0}$  belongs to  $\mathcal{D}_{\phi, F_0}$ ; see [38, hint for Problem 1.7.4]. Convergence in distribution will be understood in the sense of [28, 35].

#### 3.1. Asymptotic distributions of plug-in estimators

The quasi-Hadamard differentiability established in Theorems 2.4 and 2.7 provides a very general device to determine asymptotic distributions of plug-in estimators of law-invariant coherent risk measures. Recall that the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  was assumed to be atomless.

**Theorem 3.1** *Suppose that Assumptions 2.1 and 2.3 hold, or that Assumption 2.6 holds. Let  $\widehat{F}_n : \Omega \rightarrow D$  be a map for every  $n \in \mathbb{N}$ , and assume that the following conditions hold:*

- (a)  $\widehat{F}_n$  takes values only in  $\mathbb{F}_\mathcal{X}$  and is  $(\mathcal{F}, \mathcal{D})$  measurable,  $n \in \mathbb{N}$ .
- (b)  $\widehat{F}_n - F_0$  takes values only in  $D_{\phi, F_0}$ ,  $n \in \mathbb{N}$ .
- (c) There are some random element  $B^\circ$  of  $(D_{\phi, F_0}, \mathcal{D}_{\phi, F_0})$  as well as some  $\|\cdot\|_\phi$ -separable and  $\mathcal{D}_{\phi, F_0}$ -measurable subset  $C \subset D_{\phi, F_0}$  with  $\mathbb{P}[B^\circ \in C] = 1$ , and a nondecreasing sequence  $(r_n) \subset (0, \infty)$  with  $r_n \uparrow \infty$ , such that

$$r_n(\widehat{F}_n - F_0) \xrightarrow{d} B^\circ \quad \text{in } (D_{\phi, F_0}, \mathcal{D}_{\phi, F_0}, \|\cdot\|_\phi). \quad (12)$$

Then

$$r_n(\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F_0)) \xrightarrow{d} \dot{\mathcal{R}}_{\rho, F_0}(B^\circ) \quad \text{in } (\mathbb{R}, \mathcal{B}(\mathbb{R})), \quad (13)$$

with  $\dot{\mathcal{R}}_{\rho, F_0}$  defined as in (7). If in addition  $\int_{-\infty}^0 g_\rho(\delta F_0(x)) dx < \infty$  holds for some  $\delta \in (0, 1)$  (which, e.g., is the case if the restriction of  $F_0 \phi$  to  $(-\infty, 0)$  is bounded), then  $\dot{\mathcal{R}}_{\rho, F_0}$  is as in (8).

**Proof** First of all we note that  $\widehat{F}_n - F_0$  is  $(\mathcal{F}, \mathcal{D}_{\phi, F_0})$ -measurable, because we assumed that  $\widehat{F}_n$  is  $(\mathcal{F}, \mathcal{D})$ -measurable,  $\widehat{F}_n - F_0 \in D_{\phi, F_0}$  and  $\mathcal{D}_{\phi, F_0} = \mathcal{D} \cap D_{\phi, F_0}$ . In particular,  $r_n(\widehat{F}_n - F_0)$  is a random element of  $(D_{\phi, F_0}, \mathcal{D}_{\phi, F_0})$ .

If Assumptions 2.1 and 2.3 are satisfied, then Theorem 2.4 yields that  $\mathcal{R}_\rho$  is quasi-Hadamard differentiable at  $F_0$  tangentially to  $C_{\phi, F_0} \langle D_{\phi, F_0} \rangle$ . If Assumption 2.6 holds, then  $\rho$  is a distortion risk measure and we can apply Theorem 2.7 to obtain the same conclusion as before. In particular, in both cases  $\mathcal{R}_\rho$  is quasi-Hadamard differentiable at  $F_0$  tangentially to  $C \langle D_{\phi, F_0} \rangle$ . So the claim of Theorem 3.1 would follow from the Modified Functional Delta-Method given in [8, Theorem 4.1] if we can show that condition (c) in Theorem 4.1 in [8] holds. The latter condition requires

that  $\omega' \mapsto \mathcal{R}_\rho(W(\omega') + F_0)$  is  $(\mathcal{F}', \mathcal{B}(\mathbb{R}))$ -measurable whenever  $W$  is a measurable map from some measurable space  $(\Omega', \mathcal{F}')$  to  $(D_{\phi, F_0}, \mathcal{D}_{\phi, F_0})$  such that  $W(\omega') + F_0 \in \mathbb{F}_\mathcal{X}$  for all  $\omega' \in \Omega'$ . This condition ensures in particular that left-hand side in (13) is a real-valued random variable.

Since  $W$  is  $(\mathcal{F}', \mathcal{D}_{\phi, F_0})$ -measurable and  $\mathcal{D}_{\phi, F_0}$  is the projection  $\sigma$ -field, we obtain in particular  $(\mathcal{F}', \mathcal{B}(\mathbb{R}))$ -measurability of  $\omega' \mapsto W(x, \omega')$  for every  $x \in \mathbb{R}$ . Since  $x \mapsto W(x, \omega')$  is right-continuous for every  $\omega'$ , the mapping  $(x, \omega') \mapsto W(x, \omega')$  is  $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$ -measurable. In particular, the same is true for the mapping  $(x, \omega') \mapsto W(x, \omega') + F_0(x)$ . In view of

$$\mathcal{R}_g(F) = \int_{-\infty}^0 g(F(x)) dx - \int_0^\infty (1 - g(F(x))) dx \quad \text{for all } F \in \mathbb{F}_\mathcal{X}, g \in \mathcal{G}_\rho,$$

we may conclude from Fubini's theorem that the mapping  $\omega \mapsto \mathcal{R}_g(W(\omega') + F_0)$  is  $(\mathcal{F}', \mathcal{B}(\mathbb{R}))$ -measurable for every  $g \in \mathcal{G}_\rho$ . Moreover, for any  $F \in \mathbb{F}_\mathcal{X}$ , the mapping  $g \mapsto \mathcal{R}_g(F)$  on  $\mathcal{G}_\rho$  is lower semicontinuous w.r.t. the uniform metric by Theorem 2.2 (v). Recalling compactness of  $\mathcal{G}_\rho$ , there is some countable subset  $\mathcal{G}_0 \subset \mathcal{G}_\rho$  such that  $\mathcal{R}_\rho(F) = \sup_{g \in \mathcal{G}_0} \mathcal{R}_g(F)$  holds for every  $F \in \mathbb{F}_\mathcal{X}$ . Hence, the mapping  $\omega' \mapsto \mathcal{R}_\rho(W(\omega') + F_0)$  is  $(\mathcal{F}', \mathcal{B}(\mathbb{R}))$ -measurable. This completes the proof.  $\square$

**Remark 3.2** The preceding theorem is related as follows to previously obtained results in the literature. In the special case of a distortion risk measure associated with a (possibly nonconcave) distortion function  $g$  whose right-sided derivative  $g'$  is bounded, the result of Theorem 3.1 also follows from Theorem 2.5 in [8]. When, on the other hand,  $\rho$  is a general law-invariant coherent risk measure and  $\widehat{F}_n$  are the empirical distribution functions of a strongly mixing stationary sequences of random variables with exponential decay of the mixing coefficients, the asymptotic distributions of plug-in estimators have been derived in Theorems 3.1 and 4.1 in [6] under certain integrability conditions.  $\diamond$

Notice that the subset  $C_{0, \phi, F_0} \subset C_{\phi, F_0}$  of all functions  $v \in C_{\phi, F_0}$  satisfying  $\lim_{x \rightarrow \pm\infty} v(x) = 0$  is  $\|\cdot\|_\phi$ -separable and  $\mathcal{D}_{\phi, F_0}$ -measurable, as required by condition (c) in Theorem 3.1. For details see Corollary B.4 in the Appendix B. Of course, we have  $C_{0, \phi, F_0} = C_{\phi, F_0}$  when  $\lim_{x \rightarrow \pm\infty} \phi(x) = \infty$ . The following Examples 3.3–3.5 illustrate condition (c) of Theorem 3.1, where  $\widehat{F}_n$  will always be the empirical distribution function  $\widehat{F}_n := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{[X_i, \infty)}$  of the first  $n$  variables of a sequence  $X_1, X_2, \dots$  of identically distributed random variables on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

**Example 3.3** (*Independent data*) Assume that  $X_1, X_2, \dots$  are i.i.d. with distribution function  $F_0$ , and let  $\phi$  be a weight function. If  $\int \phi^2 dF_0 < \infty$ , then Theorem 6.2.1 in [35] shows that for the empirical distribution function  $\widehat{F}_n$  of  $X_1, \dots, X_n$

$$\sqrt{n}(\widehat{F}_n - F_0) \xrightarrow{d} B_{F_0}^\circ \quad (\text{in } (D_{\phi, F_0}, \mathcal{D}_{\phi, F_0}, \|\cdot\|_\phi)),$$

where  $B_{F_0}^\circ$  is an  $F_0$ -Brownian bridge, i.e. a centered Gaussian process with covariance function  $\Gamma(y_0, y_1) = F_0(y_0 \wedge y_1)(1 - F_0(y_0 \vee y_1))$ . Notice that  $B_{F_0}^\circ$  jumps where  $F_0$  jumps, and that  $\lim_{x \rightarrow \pm\infty} B_{F_0}^\circ(x) = 0$ . Thus,  $B_{F_0}^\circ$  takes values only in the  $\|\cdot\|_\phi$ -separable and  $\mathcal{D}_{\phi, F_0}$ -measurable subset  $C_{0, \phi, F_0}$  of  $C_{\phi, F_0}$ .  $\diamond$

**Example 3.4** (*Weakly dependent data*) Let  $(X_i)$  be strictly stationary and  $\alpha$ -mixing with mixing coefficients satisfying  $\alpha(n) = \mathcal{O}(n^{-\theta})$  for some  $\theta > 1 + \sqrt{2}$ . Let  $F_0$  be the distribution function of the  $X_i$ , let  $\lambda \geq 0$ , and set  $\phi_\lambda(x) := (1 + |x|)^\lambda$ . If  $F_0$  is continuous and has a finite  $\gamma$ -moment for some  $\gamma > 2\theta\lambda/(\theta - 1)$ , then it can easily be deduced from Theorem 2.2 in [33] that for the empirical distribution function  $\widehat{F}_n$  of  $X_1, \dots, X_n$

$$\sqrt{n}(\widehat{F}_n - F_0) \xrightarrow{d} \widetilde{B}_{F_0}^\circ \quad (\text{in } (D_{\phi_\lambda, F_0}, \mathcal{D}_{\phi_\lambda, F_0}, \|\cdot\|_{\phi_\lambda}))$$

with  $\widetilde{B}_F^\circ$  a continuous centered Gaussian process with covariance function  $\Gamma(y_0, y_1) = F_0(y_0 \wedge y_1)(1 - F_0(y_0 \vee y_1)) + \sum_{i=0}^1 \sum_{k=2}^\infty \text{Cov}(\mathbb{1}_{\{X_1 \leq y_i\}}, \mathbb{1}_{\{X_k \leq y_{1-i}\}})$ . See also [8, Section 3.3]. Notice that  $\widetilde{B}_{F_0}^\circ$  takes values only in the  $\|\cdot\|_\phi$ -separable and  $\mathcal{D}_{\phi, F_0}$ -measurable subset of all continuous functions within  $D_{\phi, F_0}$ . Also notice that under some mild regularity conditions, strictly stationary GARCH( $p, q$ ) processes are  $\alpha$ -mixing with  $\alpha(n) \leq c \varrho^n$ ,  $n \in \mathbb{N}$ , for some constants  $c > 0$  and  $\varrho \in (0, 1)$ ; cf. [25]. Thus, these GARCH processes always satisfy the above mentioned assumption on  $(\alpha(n))$ . If  $(X_i)$  is even  $\beta$ - or  $\rho$ -mixing, then the above mixing condition can be relaxed; cf. [2, 33].  $\diamond$

**Example 3.5** (*Strongly dependent data*) Consider the linear process  $X_t := \sum_{s=0}^\infty a_s \varepsilon_{t-s}$ ,  $t \in \mathbb{N}$ , where  $(\varepsilon_i)_{i \in \mathbb{Z}}$  are i.i.d. random variables on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with zero mean and finite variance, and the coefficients  $a_s$  satisfy  $\sum_{s=0}^\infty a_s^2 < \infty$ . Then  $X_1, X_2, \dots$  are identically distributed  $L^2(\Omega, \mathcal{F}, \mathbb{P})$  random variables. Further, let  $\lambda \geq 0$  and assume that the following assertions hold:

- (i)  $a_s = s^{-\beta} \ell(s)$ ,  $s \in \mathbb{N}$ , where  $\beta \in (\frac{1}{2}, 1)$  and  $\ell$  is slowly varying at infinity.
- (ii)  $\mathbb{E}[|\varepsilon_0|^{2+2\lambda}] < \infty$ .
- (iii) The distribution function  $G$  of  $\varepsilon_0$  is twice differentiable and satisfies the integrability condition  $\sum_{j=1}^2 \int |G^{(j)}(x)|^2 \phi_{2\lambda}(x) dx < \infty$ .

Under these conditions the covariances  $\text{Cov}(X_1, X_t)$  are *not* summable over  $t \in \mathbb{N}$  and thus the process exhibits strong dependence (long-memory). For instance, the infinite moving average representation of an ARFIMA( $p, d, q$ ) process with fractional difference parameter  $d \in (0, 1/2)$  satisfies assumption (a) with  $\beta = 1 - d$ ; see, for instance, [20, Section 3]. It is shown in [7, Theorem 2.1] that for the distribution function  $F_0$  of the  $X_i$ , the empirical distribution function  $\widehat{F}_n$  of  $X_1, \dots, X_n$  and  $\phi_\lambda(x) := (1 + |x|)^\lambda$

$$n^{\beta-1/2} \ell(n)^{-1} (\widehat{F}_n(\cdot) - F_0(\cdot)) \xrightarrow{d} c_{1,\beta} f_0(\cdot) Z \quad (\text{in } (D_{\phi_\lambda, F_0}, \mathcal{D}_{\phi_\lambda, F_0}, \|\cdot\|_{\phi_\lambda})),$$

where  $f_0$  is the Lebesgue density of  $F_0$ ,  $Z$  is a standard normally distributed random variable, and  $c_{1,\beta} := \{\mathbb{E}[\varepsilon_0^2](1 - (\beta - \frac{1}{2}))(1 - (2\beta - 1))/(\int_0^\infty (x + x^2)^{-\beta} dx)\}^{1/2}$ . Notice that condition (iii) ensures that the distribution function  $F_0$  of  $X_1$  is differentiable with derivative  $f_0 \in D_{\phi_\lambda}$ ; cf. inequality (30) in [40] with  $n = \infty$ ,  $\kappa = 1$  and  $\gamma = 2\lambda$ . Also notice that the limiting process  $c_{1,\beta} f_0(\cdot) Z$  takes values only in the  $\|\cdot\|_\phi$ -separable and  $\mathcal{D}_{\phi, F_0}$ -measurable subset of all continuous functions within  $D_{\phi, F_0}$ .  $\diamond$

### 3.2. Strong laws for plug-in estimators

The following theorem generalizes the result of Section 3.2 in [41].

**Theorem 3.6** *Suppose that Assumptions 2.1 and 2.3 hold, or that Assumption 2.6 holds. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space,  $\widehat{F}_n : \Omega \rightarrow D$  be a map for every  $n \in \mathbb{N}$ , and assume that the following conditions hold:*

- (a)  $\widehat{F}_n$  takes values only in  $\mathbb{F}_\mathcal{X}$  and is  $(\mathcal{F}, \mathcal{D})$  measurable,  $n \in \mathbb{N}$ .
- (b)  $\widehat{F}_n - F_0$  takes values only in  $D_{\phi, F_0}$ ,  $n \in \mathbb{N}$ .
- (c) There is some nondecreasing sequence  $(r_n) \subset (0, \infty)$  such that

$$r_n \|\widehat{F}_n - F_0\|_\phi \longrightarrow 0 \quad \mathbb{P}\text{-a.s.} \quad (14)$$

Then

$$r_n (\mathcal{R}_\rho(\widehat{F}_n) - \mathcal{R}_\rho(F_0)) \longrightarrow 0 \quad \mathbb{P}\text{-a.s.} \quad (15)$$

**Proof** Notice that all involved expressions are measurable; cf. the proof of Theorem 3.1. Assumptions 2.1 and 2.3 (or Assumption 2.6) and Theorem 2.4 ensure that  $\mathcal{R}_\rho$  is quasi-Hadamard differentiable at  $F_0$  tangentially to  $C_{\phi, F_0} \langle D_{\phi, F_0} \rangle$ . By Lemma A.5, we may conclude that  $\mathcal{R}_\rho$  is also quasi-Lipschitz continuous at  $F_0$  along  $D_{\phi, F_0}$  in the sense of Definition A.3. Thus, (14) obviously implies (15).  $\square$

The following Examples 3.7–3.8 illustrate condition (c) of Theorem 3.6, where  $\widehat{F}_n$  will always be the empirical distribution function  $\widehat{F}_n := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{[X_i, \infty)}$  of the first  $n$  variables of a sequence  $X_1, X_2, \dots$  of identically distributed random variables on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

**Example 3.7** (*Independent data*) Assume that  $X_1, X_2, \dots$  are i.i.d. with distribution function  $F_0$ . Let  $\phi$  be any weight function, and  $r \in [0, \frac{1}{2})$ . If the sequence  $(X_i)$  is i.i.d. and  $\int \phi^{1/(1-r)} dF < \infty$ , then (14) holds for  $r_n = n^r$ . This is an immediate consequence of Theorem 7.3 in [1]; cf. Theorem 2.1 in [41].  $\diamond$

**Example 3.8** (*Weakly dependent data*) Suppose that  $\int \phi dF < \infty$ . Further suppose that  $(X_i)$  is  $\alpha$ -mixing with mixing coefficients  $\alpha(n)$ , let  $\alpha(t) := \alpha(\lfloor t \rfloor)$  be the càdlàg extension of  $\alpha(\cdot)$  from  $\mathbb{N}$  to  $\mathbb{R}_+$ , and assume that  $\int_0^1 \log(1 + \alpha^\rightarrow(s/2)) \overline{G}^\rightarrow(s) ds < \infty$  for  $\overline{G} := 1 - G$ , where  $G$  denotes the df of  $\phi(X_1)$  and  $\overline{G}^\rightarrow$  the right-continuous inverse of  $\overline{G}$ . It was shown in [41, Theorem 2.3] that, under the imposed assumptions, (14) holds for  $r_n = 1$ . Notice that the integrability condition above holds in particular if  $\mathbb{E}[\phi(X_1) \log^+ \phi(X_1)] < \infty$  and  $\alpha(n) = \mathcal{O}(n^{-\vartheta})$  for some arbitrarily small  $\vartheta > 0$ ; cf. [30, Application 5, p. 924].

Suppose that the sequence  $(X_i)$  is  $\alpha$ -mixing with mixing coefficients  $\alpha(n)$ . Let  $r \in [0, \frac{1}{2})$  and assume that  $\alpha(n) \leq Kn^{-\vartheta}$  for all  $n \in \mathbb{N}$  and some constants  $K > 0$  and  $\vartheta > 2r$ . Then (14) holds for  $\phi \equiv 1$  and  $r_n = n^r$ ; cf. [41, Theorem 2.2].  $\diamond$

## 4. Proofs

### 4.1. Proof of Theorem 2.2

Assertions (i), (ii) and (iv) are known from Proposition 5.1 in [6]; see also independent proof of (ii) and (iv) in [23]. Moreover it follows from the last calculation in Section 4.3 of [23] that  $|g(s) - g(t)| \leq g_\rho(|s - t|)$  for every  $g \in \mathcal{G}_\rho$  and  $s, t \in [0, 1]$ . This implies assertion (iii), because all elements of  $\mathcal{G}_\rho$  are concave. Thus it remains to show assertion (v).

For this purpose, let  $(g_k)$  denote any sequence in  $\mathcal{G}_\rho$  which converges to some  $g \in \mathcal{G}_\rho$  w.r.t. the uniform metric, in particular  $g_k(F_0(x)) \rightarrow g(F_0(x))$  and therefore also  $1 - g_k(F_0(x)) \rightarrow 1 - g(F_0(x))$  for every  $x \in \mathbb{R}$ . Then by Fatou's lemma

$$\liminf_{k \rightarrow \infty} \int_{-\infty}^0 g_k(F_0(x)) dx \geq \int_{-\infty}^0 g(F_0(x)) dx. \quad (16)$$

Furthermore, for any  $x > F_0^{\leftarrow}(1/2)^+$ , concavity of  $g_k$  along with (iii) implies

$$1 - g_k(F_0(x)) \leq (1 - F_0(x))g'_k(F_0(x)) \leq (1 - F_0(x))g'_k(1/2) \leq (1 - F_0(x))4g_\rho(1/4).$$

This means

$$|(1 - g_k(F_0))\mathbb{1}_{[0, \infty)}| \leq (1 - g_k(F_0))\mathbb{1}_{[0, F_0^{\leftarrow}(1/2)^+]} + 4g_\rho(1/4)(1 - F_0)\mathbb{1}_{(F_0^{\leftarrow}(1/2)^+, \infty)} \quad (17)$$

Recall that  $F_0$  is the distribution function of some  $\mathbb{P}$ -integrable random variable so that  $\int_0^\infty (1 - F_0(x)) dx < \infty$ . Since in addition  $(1 - g_k(F_0))\mathbb{1}_{[0, F_0^{\leftarrow}(1/2)^+]}$  is bounded, it follows that the right-hand side of (17) is integrable w.r.t. the Lebesgue measure on  $\mathbb{R}$ . Hence in view of (17), the application of the Dominated Convergence Theorem yields

$$\lim_{k \rightarrow \infty} \int_0^\infty (1 - g_k(F_0(x))) dx = \int_0^\infty (1 - g(F_0(x))) dx. \quad (18)$$

Combining (16) and (18), we may conclude  $\liminf_{k \rightarrow \infty} \mathcal{R}_{g_k}(F_0) \geq \mathcal{R}_g(F_0)$ .

Let us now suppose that in addition  $\int_{-\infty}^0 g_\rho(\gamma F_0(x)) dx < \infty$  holds for some  $\gamma \in (0, 1)$ . By concavity of  $g_k$  along with (ii), we obtain for  $x < 0$

$$g_k(x) \leq \frac{g_k(\gamma F_0(x))}{\gamma} \leq \frac{g_\rho(\gamma F_0(x))}{\gamma}.$$

Then we may have  $\lim_{k \rightarrow \infty} \int_{-\infty}^0 g_k(F_0(x)) dx = \int_{-\infty}^0 g(F_0(x)) dx$  by the Dominated Convergence Theorem. Due to (18), this implies  $\lim_{k \rightarrow \infty} \mathcal{R}_{g_k}(F_0) = \mathcal{R}_g(F_0)$ , completing the proof.

### 4.2. Auxiliary lemma

**Lemma 4.1** *Under the Assumptions 2.1 and 2.3, the following assertions hold:*

- (i) *For every  $v \in C_{\phi, F_0}$ , the mapping  $\mathcal{G}_\rho \rightarrow \mathbb{R}$ ,  $g \mapsto \int_{F \rightarrow (0)}^{F^{\leftarrow}(1)} g'(F(x)) v(x) dx$  is continuous w.r.t. the uniform metric on  $\mathcal{G}_\rho$ .*
- (ii) *The mapping  $D_{\phi, F_0} \rightarrow \ell^\infty(\mathcal{G}_\rho)$ ,  $v \mapsto (\int_{F \rightarrow (0)}^{F_0^{\leftarrow}(1)} g'(F_0(x)) v(x) dx)_{g \in \mathcal{G}_\rho}$  is continuous w.r.t.  $\|\cdot\|_\phi$ , where  $\ell^\infty(\mathcal{G}_\rho)$  is the space of all bounded real-valued functions on  $\mathcal{G}_\rho$  equipped with the sup-norm.*

**Proof** We will first prove assertion (i). Let  $(g_k)$  be any sequence in  $\mathcal{G}_\rho$  which converges to some  $g \in \mathcal{G}_\rho$  w.r.t. the uniform metric. In view of the Dominated Convergence Theorem, it suffices to show 1) that  $g'_k(F_0(x))v(x)$  converges to  $g'(F_0(x))v(x)$  for Lebesgue a.e.  $x \in (F_0^\rightarrow(0), F_0^\leftarrow(1))$ , and 2) that there is some Lebesgue integrable majorant for the sequence  $(g'_k(F_0(\cdot))v(\cdot))$ .

To verify condition 1), let us first fix any  $t \in (0, 1)$ . Then, on the one hand, we may find for every  $\varepsilon > 0$  some  $\delta \in (0, 1 - t)$  such that

$$g'(t) - \varepsilon < \frac{g(t + \delta) - g(t)}{\delta} = \liminf_{k \rightarrow \infty} \frac{g_k(t + \delta) - g_k(t)}{\delta} \leq \liminf_{k \rightarrow \infty} g'_k(t),$$

which means that

$$g'(t) \leq \liminf_{k \rightarrow \infty} g'_k(t). \quad (19)$$

On the other hand, we have for every  $\delta \in (0, t)$

$$\limsup_{k \rightarrow \infty} g'_k(t) \leq \limsup_{k \rightarrow \infty} \frac{g_k(t) - g_k(t - \delta)}{\delta} = \frac{g(t) - g(t - \delta)}{\delta}.$$

This implies

$$\limsup_{k \rightarrow \infty} g'_k(t) \leq \lim_{\delta \rightarrow 0+} \frac{g(t) - g(t - \delta)}{\delta} = g'_-(t), \quad (20)$$

where  $g'_-(t)$  denotes the left-sided derivative of  $g$  at  $t$ . Combining (19) and (20), we obtain

$$g'(F_0(x)) = \lim_{k \rightarrow \infty} g'_k(F_0(x)) \quad \text{for all } x \in \mathbb{R} \setminus A, \quad (21)$$

where  $A$  denotes the set consisting of all  $x \in (F_0^\rightarrow(0), F_0^\leftarrow(1))$  for which  $g'_-(F_0(x))$  differs from  $g'(F_0(x))$ . Notice that  $A$  is at most countable due to monotonicity of the left- and the right-sided derivative functions of  $g$ . Since by assumption  $F_0$  is strictly increasing on  $(F^\rightarrow(0), F^\leftarrow(1))$ , this implies  $g'(F_0(x)) = \lim_k g'_k(F_0(x))$  for all but countably many  $x$  in  $(F^\rightarrow(0), F^\leftarrow(1))$ .

To verify condition 2), let  $\gamma \in (0, 1)$  be as in the integrability condition (b) from Assumption 2.3. Then, in view of assertion (iii) from Theorem 2.2, we may conclude

$$\sup_{k \in \mathbb{N}} |g'_k(F_0(x))v(x)| \leq \frac{\sup_{k \in \mathbb{N}} |g'_k(F_0(x))| \|v\|_\phi}{\phi(x)} \leq \frac{g_\rho(\gamma F_0(x))}{F_0(x)\phi(x)} \frac{\|v\|_\phi}{\gamma} \quad (22)$$

for every  $x \in (F_0^\rightarrow(0), F_0^\leftarrow(1))$ . The integrability condition (b) from Assumption 2.3 ensures that the right-hand side in (22) is Lebesgue integrable over  $(F^\rightarrow(0), F^\leftarrow(1))$ .

Now, we will prove assertion (ii). Let  $(v_k)$  be any sequence in  $D_{\phi, F_0}$  which converges to some  $v \in D_{\phi, F_0}$  w.r.t.  $\|\cdot\|_\phi$ . We clearly have

$$\begin{aligned} & \sup_{g \in \mathcal{G}_\rho} \left| \int_{F_0^\rightarrow(0)}^{F_0^\leftarrow(1)} g'(F(x)) v_k(x) dx - \int_{F_0^\rightarrow(0)}^{F_0^\leftarrow(1)} g'(F_0(x)) v(x) dx \right| \\ & \leq \int_{F_0^\rightarrow(0)}^{F_0^\leftarrow(1)} \sup_{g \in \mathcal{G}_\rho} g'(F(x)) |v_k(x) - v(x)| dx \\ & \leq \int_{F_0^\rightarrow(0)}^{F_0^\leftarrow(1)} \frac{g_\rho(\gamma F(x))}{\gamma F(x)\phi(x)} dx \|v_k - v\|_\phi, \end{aligned}$$

and the integrability condition (b) from Assumption 2.3 ensures that the latter expression converges to 0 as  $k \rightarrow \infty$ . The proof is now complete.  $\square$

### 4.3. Proof of Theorem 2.4

The statistical functional  $\mathcal{R}_\rho : \mathbb{F}_\mathcal{X} \rightarrow \mathbb{R}$  can be represented as composition

$$\mathcal{R}_\rho = \mathcal{S}_\rho \circ \mathcal{T}_\rho$$

with  $\mathcal{T}_\rho : \mathbb{F}_\mathcal{X} \rightarrow \ell^\infty(\mathcal{G}_\rho)$  and  $\mathcal{S}_\rho : \ell^\infty(\mathcal{G}_\rho) \rightarrow \mathbb{R}$  given by

$$\mathcal{T}_\rho(F) := (\mathcal{R}_g(F))_{g \in \mathcal{G}_\rho}$$

and

$$\mathcal{S}_\rho((x_g)_{g \in \mathcal{G}_\rho}) := \sup_{g \in \mathcal{G}_\rho} x_g,$$

respectively, where  $\ell^\infty(\mathcal{G}_\rho)$  is the space of all bounded real-valued functions on  $\mathcal{G}_\rho$  equipped with the sup-norm. In the following we will show that the functional  $\mathcal{T}_\rho$  is quasi-Hadamard differentiable at  $F_0$  tangentially to  $D_{\phi, F_0} \langle D_{\phi, F_0} \rangle$  with quasi-Hadamard derivative  $\dot{\mathcal{T}}_{\rho, F_0}$  given by

$$\dot{\mathcal{T}}_{\rho, F_0}(v) := \left( \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} g'(F_0(x)) v(x) dx \right)_{g \in \mathcal{G}_\rho}, \quad v \in C_{\phi, F_0}. \quad (23)$$

This is sufficient for the proof of Theorem 2.4. Indeed, it is known from [31, Proposition 1] that the mapping  $\mathcal{S}_\rho$  is Hadamard differentiable at every  $(x_g)_{g \in \mathcal{G}_\rho}$  (tangentially to the whole space  $\ell^\infty(\mathcal{G}_\rho)$ ) with (possibly nonlinear) Hadamard derivative  $\dot{\mathcal{S}}_{\rho, (x_g)_{g \in \mathcal{G}_\rho}}$  given by

$$\dot{\mathcal{S}}_{\rho, (x_g)_{g \in \mathcal{G}_\rho}}((w_g)_{g \in \mathcal{G}_\rho}) := \lim_{\varepsilon \downarrow 0} \sup_{g \in \mathcal{G}_\rho((x_g)_{g \in \mathcal{G}_\rho}, \varepsilon)} w_g, \quad (w_g)_{g \in \mathcal{G}_\rho} \in \ell^\infty(\mathcal{G}_\rho),$$

where  $\mathcal{G}_\rho((x_g)_{g \in \mathcal{G}_\rho}, \varepsilon)$  denotes the set of all  $g \in \mathcal{G}_\rho$  satisfying  $\sup_{h \in \mathcal{G}_\rho} x_h - \varepsilon \leq x_g$ . Moreover, the restriction of  $F_0$  to  $(F_0^{\rightarrow}(0), F_0^{\leftarrow}(1))$  is injective by assumption. Therefore, in view of (i) in Theorem 2.2, we obtain for  $v \in C_{\phi, F_0}$

$$\dot{\mathcal{S}}_{\rho, (\mathcal{R}_g(F_0))_{g \in \mathcal{G}_\rho}} \left( \left( \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} g'(F_0(x)) v(x) dx \right)_{g \in \mathcal{G}_\rho} \right) = \lim_{\varepsilon \downarrow 0} \sup_{g \in \mathcal{G}_\rho(F_0, \varepsilon)} \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} g'(F_0(x)) v(x) dx$$

If in addition  $\int_{-\infty}^0 g_\rho(\delta F_0(x)) dx < \infty$  holds for some  $\delta \in (0, 1)$ , then by parts (i) and (v) of Theorem 2.2, the set  $\mathcal{G}_\rho(F_0)$  is nonempty, and

$$\dot{\mathcal{S}}_{\rho, (\mathcal{R}_g(F_0))_{g \in \mathcal{G}_\rho}} \left( \left( \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} g'(F_0(x)) v(x) dx \right)_{g \in \mathcal{G}_\rho} \right) = \sup_{g \in \mathcal{G}_\rho(F_0)} \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} g'(F_0(x)) v(x) dx$$

for every  $v \in C_{\phi, F_0}$ . Hence the full claim of Theorem 2.4 follows from the chain rule in Lemma A.2.

We are now going to establish the above mentioned quasi-Hadamard differentiability of  $\mathcal{T}_\rho$ . First of all notice that the map  $\dot{\mathcal{T}}_{\rho, F_0}$  defined in (23) is continuous w.r.t.  $\|\cdot\|_\phi$  by part (ii) of Lemma 4.1. Now, let  $(v, (v_n), (h_n))$  be a triplet with  $v \in C_{\phi, F_0}$ ,  $(v_n) \subset D_{\phi, F_0}$  satisfying  $\|v_n - v\|_\phi \rightarrow 0$  and  $(F_0 + h_n v_n) \subset \mathbb{F}_\mathcal{X}$ , and  $(h_n) \subset (0, \infty)$  satisfying  $h_n \rightarrow 0$ . We have to show that

$$\lim_{n \rightarrow \infty} \left\| \frac{\mathcal{T}_\rho(F_0 + h_n v_n) - \mathcal{T}_\rho(F_0)}{h_n} - \dot{\mathcal{T}}_{\rho, F_0}(v) \right\|_\infty = 0,$$

that is,

$$\lim_{n \rightarrow \infty} \sup_{g \in \mathcal{G}_\rho} \left| \frac{\mathcal{R}_g(F_0 + h_n v_n) - \mathcal{R}_g(F_0)}{h_n} - \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} g'(F_0(x)) v(x) dx \right| = 0 \quad (24)$$

or, in other words,

$$\lim_{n \rightarrow \infty} \sup_{g \in \mathcal{G}_\rho} \left| \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} \frac{g((F_0 + h_n v_n)(x)) - g(F_0(x))}{h_n} dx - \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} g'(F_0(x)) v(x) dx \right| = 0. \quad (25)$$

By Assumption 2.3(b), there exist  $r \in \mathbb{N}_0$  and  $F_0^{\rightarrow}(0) =: a_0 < a_1 < \dots < a_{r+1} := F_0^{\leftarrow}(1)$  such that for every  $i = 1, \dots, r$  the restriction  $F_0|_{(a_{i-1}, a_i)}$  of  $F_0$  to  $(a_{i-1}, a_i)$  is continuously differentiable with strictly positive derivative. For every  $i = 0, \dots, r$ , we consider the extension  $F_{0,i} := F_0 \mathbb{1}_{[a_i, a_{i+1})} + \mathbb{1}_{[a_{i+1}, \infty)}$  of  $F_0|_{(a_i, a_{i+1})}$  from  $(a_i, a_{i+1})$  to  $\mathbb{R}$ . This extension  $F_{0,i}$  is contained in  $\mathbb{F}_\mathcal{X}$ . Indeed, if  $X_0$  is any random variable from  $\mathcal{X}$  with distribution function  $F_0$ , then  $X_{0,i} := a_i \vee (X_0 \wedge a_{i+1})$  belongs to  $\mathcal{X}$  (recall that  $\mathcal{X}$  is a Stonean vector lattice) and  $F_{0,i}$  is the distribution function of  $X_{0,i}$ . Further, for every  $g \in \mathcal{G}_\rho$  we have that

$$\begin{aligned} & \left| \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} \frac{g((F_0 + h_n v_n)(x)) - g(F_0(x))}{h_n} dx - \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} g'(F_0(x)) v(x) dx \right| \\ & \leq \sum_{i=0}^r \left| \int_{F_{0,i}^{\rightarrow}(0)}^{F_{0,i}^{\leftarrow}(1)} \frac{g((F_{0,i} + h_n v_{n,i})(x)) - g(F_{0,i}(x))}{h_n} dx - \int_{F_{0,i}^{\rightarrow}(0)}^{F_{0,i}^{\leftarrow}(1)} g'(F_{0,i}(x)) v_i(x) dx \right|, \end{aligned}$$

with  $v_{n,i} := v_n \mathbb{1}_{[a_i, a_{i+1})}$  and  $v_i := v \mathbb{1}_{[a_i, a_{i+1})}$ . Thus, for (25) it suffices to show that

$$\lim_{n \rightarrow \infty} \sup_{g \in \mathcal{G}_\rho} \left| \int_{F_{0,i}^{\rightarrow}(0)}^{F_{0,i}^{\leftarrow}(1)} \frac{g((F_{0,i} + h_n v_{n,i})(x)) - g(F_{0,i}(x))}{h_n} dx - \int_{F_{0,i}^{\rightarrow}(0)}^{F_{0,i}^{\leftarrow}(1)} g'(F_{0,i}(x)) v(x) dx \right| = 0$$

for every  $i = 1, \dots, r$ . So, since  $v_{n,i} \in D_{\phi, F_{0,i}}$ ,  $v_i \in C_{\phi, F_{0,i}}$ , and the restriction of  $F_{0,i}$  to  $(F_{0,i}^{\rightarrow}(0), F_{0,i}^{\leftarrow}(1))$  is continuously differentiable with strictly positive derivative, we may without loss of generality restrict ourselves to the case  $r = 0$ . In the remainder of the proof we will show (25) for  $r = 0$ .

Let  $(t_k)$  be any sequence in  $(0, 1/2)$  with  $t_k \downarrow 0$ . Moreover, let the map  $g_k : [0, 1] \rightarrow [0, 1]$  be defined by

$$g_k(t) := \int_0^t \mathbb{1}_{[t_k, 1-t_k]}(s) g'(s) ds, \quad t \in [0, 1], \quad (26)$$

and notice that  $g(t) = \lim_{k \rightarrow \infty} g_k(t)$  for every  $t \in [0, 1]$ . Then, if we set

$$a_n(g) := \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} \frac{g((F_0 + h_n v_n)(x)) - g(F_0(x))}{h_n} dx, \quad (27)$$

$$a_{n,k}(g) := \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} \frac{g_k((F_0 + h_n v_n)(x)) - g_k(F_0(x))}{h_n} dx \quad (28)$$

and

$$b(g) := \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} g'(F_0(x)) v(x) dx, \quad (29)$$

$$b_k(g) := \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} g'(F_0(x)) v(x) \mathbb{1}_{[t_k, 1-t_k]}(F_0(x)) dx. \quad (30)$$



Of course, (25) follow if we can show that

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{g \in \mathcal{G}_\rho} |a_n(g) - a_{n,k}(g)| = 0, \quad (31)$$

$$\lim_{n \rightarrow \infty} \sup_{g \in \mathcal{G}_\rho} |a_{n,k}(g) - b_k(g)| = 0 \quad \text{for all } k \in \mathbb{N}, \quad (32)$$

$$\lim_{k \rightarrow \infty} \sup_{g \in \mathcal{G}_\rho} |b_k(g) - b(g)| = 0. \quad (33)$$

We will now verify in Steps 1–3 that (31)–(33) hold true.

*Step 1.* We first show (33). We clearly have

$$\sup_{g \in \mathcal{G}_\rho} |b_k(g) - b(g)| \leq \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} \sup_{g \in \mathcal{G}_\rho} g'(F_0(x)) |v(x)| \mathbb{1}_{(0,t_k) \cup (1-t_k,1)}(F_0(x)) dx,$$

and the latter integrand converges to zero as  $k \rightarrow \infty$  for every  $x \in (F_0^{\rightarrow}(0), F_0^{\leftarrow}(1))$ . By part (iii) of Theorem 2.2 we also have

$$\sup_{g \in \mathcal{G}_\rho} g'(F_0(x)) |v(x)| \mathbb{1}_{(0,t_k) \cup (1-t_k,1)}(F_0(x)) \leq \frac{g_\rho(\tilde{\gamma} F_0(x))}{\tilde{\gamma} F_0(x)} |v(x)| \leq \frac{g_\rho(\tilde{\gamma} F_0(x))}{\tilde{\gamma} F_0(x) \phi(x)} \|v\|_\phi$$

for every  $x \in (F_0^{\rightarrow}(0), F_0^{\leftarrow}(1))$  and all  $\tilde{\gamma} \in (0,1)$ , and so the integrability condition (b) in Assumption 2.3 ensures that we may apply the Dominated Convergence Theorem to obtain (33).

*Step 2.* We next show (32). According to Lemma A.1 in [6] we have

$$\begin{aligned} a_{n,k}(g) &= \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} \frac{g_k((F_0 + h_n v_n)(x)) - g_k(F_0(x))}{h_n} dx \\ &= \int_0^1 \frac{(F_0 + h_n v_n)^{\leftarrow}(t) - F_0^{\leftarrow}(t)}{h_n} dg_k(t) \\ &= \int_0^1 \frac{(F_0 + h_n v_n)^{\leftarrow}(s) - F_0^{\leftarrow}(s)}{h_n} \mathbb{1}_{[t_k, 1-t_k]}(t) g'(t) dt. \end{aligned}$$

By a change-of-variable  $t := F_0(x)$  and Assumption 2.3 (a), we also have

$$\begin{aligned} b_k(g) &= \int_{F_0^{\rightarrow}(0)}^{F_0^{\leftarrow}(1)} g'(F_0(x)) v(x) \mathbb{1}_{[t_k, 1-t_k]}(F_0(x)) dx \\ &= \int_0^1 g'(t) v(F_0^{\leftarrow}(t)) \mathbb{1}_{[t_k, 1-t_k]}(t) \frac{1}{F_0'(F_0^{\leftarrow}(t))} dt. \end{aligned}$$

Thus, using part (iii) of Theorem 2.2, we have

$$\begin{aligned} &\sup_{g \in \mathcal{G}_\rho} |a_{n,k}(g) - b_k(g)| \\ &\leq \sup_{g \in \mathcal{G}_\rho} \int_0^1 \left| \frac{(F_0 + h_n v_n)^{\leftarrow}(t) - F_0^{\leftarrow}(t)}{h_n} - \frac{v(F_0^{\leftarrow}(t))}{F_0'(F_0^{\leftarrow}(t))} \right| \mathbb{1}_{[t_k, 1-t_k]}(t) g'(t) dt \\ &\leq \sup_{t \in [t_k, 1-t_k]} \left| \frac{(F_0 + h_n v_n)^{\leftarrow}(t) - F_0^{\leftarrow}(t)}{h_n} - \frac{v(F_0^{\leftarrow}(t))}{F_0'(F_0^{\leftarrow}(t))} \right| \int_{t_k}^{1-t_k} \sup_{g \in \mathcal{G}_\rho} g'(t) dt \\ &\leq C_k \sup_{t \in [t_k, 1-t_k]} \left| \frac{(F_0 + h_n v_n)^{\leftarrow}(t) - F_0^{\leftarrow}(t)}{h_n} - \frac{v(F_0^{\leftarrow}(t))}{F_0'(F_0^{\leftarrow}(t))} \right| \end{aligned}$$

for  $C_k := \int_{t_k}^{1-t_k} \frac{g_\rho(\tilde{\gamma}t)}{\tilde{\gamma}t} dt < \infty$ . Now, part (i) of Lemma 21.4 in [37] yields that the latter expression converges to zero as  $n \rightarrow \infty$  for every fixed  $k \in \mathbb{N}$ .

*Step 3.* Finally we will show (31). Let  $I_n(x)$  denote the closed interval with boundary points  $F_0(x)$  and  $(F_0 + h_n v_n)(x)$ . Then we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sup_{g \in \mathcal{G}_\rho} |a_n(g) - a_{n,k}(g)| \\ & \leq \limsup_{n \rightarrow \infty} \int \mathbb{1}_{(F_0^\rightarrow(0), F_0^\leftarrow(1))}(x) \sup_{g \in \mathcal{G}_\rho} \int_{I_n(x)} \frac{\mathbb{1}_{[0, t_k] \cup [1-t_k, 1]}(s) g'(s)}{h_n} ds dx. \end{aligned} \quad (34)$$

Notice that the integrand of the  $dx$ -integral, i.e.

$$G_{n,k}(x) := \mathbb{1}_{(F_0^\rightarrow(0), F_0^\leftarrow(1))}(x) \sup_{g \in \mathcal{G}_\rho} \int_{I_n(x)} \frac{\mathbb{1}_{[0, t_k] \cup [1-t_k, 1]}(s) g'(s)}{h_n} ds, \quad x \in \mathbb{R}, \quad (35)$$

is clearly nonnegative, and measurable w.r.t. the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R})$  because its restriction to  $\mathbb{R} \setminus \mathbb{D}_n$  (with  $\mathbb{D}_n$  the set of discontinuity points of  $v_n$ ) is lower semi-continuous w.r.t. the relative topology. In Step 4 below we will show that the integrand  $G_{n,k}$  is bounded above by the nonnegative and  $\mathcal{B}(\mathbb{R})$ -measurable function

$$G(x) := \left( \frac{\|v\|_\phi + c}{\gamma} \right) \frac{g_\rho(\gamma F_0(x))}{F_0(x) \phi(x)}, \quad x \in \mathbb{R}, \quad (36)$$

where  $c \in (0, \infty)$  is some suitable constant being independent of  $n$  (and  $k$ ), and  $\gamma$  is as in condition (b) of Assumption 2.3. By condition (b) of Assumption 2.3, the mapping  $G$  is even integrable w.r.t. the Lebesgue measure on  $\mathbb{R}$ . So, applying Fatou's lemma to the sequence  $(G - G_{n,k})_n$ , we obtain from (34)

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sup_{g \in \mathcal{G}_\rho} |a_n - a_{n,k}(g)| \\ & \leq \int \mathbb{1}_{(F_0^\rightarrow(0), F_0^\leftarrow(1))}(x) \limsup_{n \rightarrow \infty} \sup_{g \in \mathcal{G}_\rho} \int_{I_n(x)} \frac{\mathbb{1}_{[0, t_k] \cup [1-t_k, 1]}(s) g'(s)}{h_n} ds dx. \end{aligned} \quad (37)$$

Step 4 below also shows that  $G$  defined in (36) provides a  $\mathcal{B}(\mathbb{R})$ -measurable majorant of the integrand of the latter  $dx$ -integral, i.e. of

$$G_k(x) := \mathbb{1}_{(F_0^\rightarrow(0), F_0^\leftarrow(1))}(x) \limsup_{n \rightarrow \infty} \sup_{g \in \mathcal{G}_\rho} \int_{I_n(x)} \frac{\mathbb{1}_{[0, t_k] \cup [1-t_k, 1]}(s) g'(s)}{h_n} ds, \quad x \in \mathbb{R},$$

and by the integrability condition (b) in Assumption 2.3 the majorant  $G$  is also  $dx$ -integrable. So, in view of the Dominated Convergence Theorem, it remains to show that  $G_k(x)$  converges to zero as  $k \rightarrow \infty$  for  $dx$ -almost all  $x \in \mathbb{R}$ . To do so, let  $x \in (F_0^\rightarrow(0), F_0^\leftarrow(1))$ . Then, by part (iii) of Theorem 2.2 and a change-of-variable  $y := F_0^{-1}(s)$  along with Assumption 2.3 (a), we obtain

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sup_{g \in \mathcal{G}_\rho} \int_{I_n(x)} \frac{\mathbb{1}_{[0, t_k] \cup [1-t_k, 1]}(s) g'(s)}{h_n} ds \\ & \leq \limsup_{n \rightarrow \infty} \int_0^1 \mathbb{1}_{I_n(x)}(s) \mathbb{1}_{[0, t_k] \cup [1-t_k, 1]}(s) \frac{g_\rho(\gamma s)}{\gamma s h_n} ds \\ & = \limsup_{n \rightarrow \infty} \int_{F_0^\rightarrow(0)}^{F_0^\leftarrow(1)} \mathbb{1}_{I_n(x)}(F_0(y)) \mathbb{1}_{[0, t_k] \cup [1-t_k, 1]}(F_0(y)) \frac{g_\rho(\gamma F_0(y))}{\gamma F_0(y) h_n} F_0'(y) dy. \end{aligned}$$

Now, if  $k$  is sufficiently large so that  $F_0(x) \in (t_k, 1 - t_k)$ , then also  $I_n(x) \subset (t_k, 1 - t_k)$  for  $n$  sufficiently large. That is, for sufficiently large  $k$  we have that the latter expression equals zero. This implies  $G_k(x) \rightarrow 0$  as  $k \rightarrow \infty$  for all  $x \in \mathbb{R}$ .

*Step 4.* Let  $G_{n,k}$  and  $G$  be defined as in (35) and (36), respectively. It remains to show that  $G_{n,k} \leq G$ . By the concavity of all  $g \in \mathcal{G}_\rho$  and part (iii) of Theorem 2.2 we obtain for every  $x \in (F_0^\rightarrow(0), F_0^\leftarrow(1))$  with  $(F_0 + h_n v_n)(x) > 0$  and  $v_n(x) \geq 0$

$$\begin{aligned}
\sup_{g \in \mathcal{G}_\rho} \int_{I_n(x)} \frac{\mathbb{1}_{[0, t_k] \cup [1 - t_k, 1]}(s) g'(s)}{h_n} ds &\leq \frac{\sup_{g \in \mathcal{G}_\rho} g'(F_0(x))}{h_n} h_n v_n(x) \\
&\leq \frac{g_\rho(\gamma F_0(x)) v_n(x) \phi(x)}{\gamma F_0(x) \phi(x)} \\
&\leq \|v_n\|_\phi \frac{g_\rho(\gamma F_0(x))}{\gamma F_0(x) \phi(x)} \\
&\leq \left( \frac{\|v\|_\phi + c}{\gamma} \right) \frac{g_\rho(\gamma F_0(x))}{F_0(x) \phi(x)} \tag{38}
\end{aligned}$$

for some suitable constant  $c \in (0, \infty)$  being independent of  $n \in \mathbb{N}$ , and  $\gamma$  as in condition (b) of Assumption 2.3. For every  $x \in (F_0^\rightarrow(0), F_0^\leftarrow(1))$  with  $(F_0 + h_n v_n)(x) > 0$  and  $v_n(x) < 0$  we further obtain by the concavity of all  $g \in \mathcal{G}_\rho$  and part (ii) of Theorem 2.2

$$\begin{aligned}
\sup_{g \in \mathcal{G}_\rho} \int_{I_n(x)} \frac{\mathbb{1}_{[0, t_k] \cup [1 - t_k, 1]}(s) g'(s)}{h_n} ds &\leq \sup_{g \in \mathcal{G}_\rho} \int_{(F_0 + h_n v_n)(x)}^{F_0(x)} \frac{g'(s)}{h_n} ds \\
&= \sup_{g \in \mathcal{G}_\rho} \frac{g(F_0(x)) - g((F_0 + h_n v_n)(x))}{h_n} \\
&\leq \sup_{g \in \mathcal{G}_\rho} \frac{g(F_0(x)) - \frac{(F_0 + h_n v_n)(x)}{F_0(x)} g(F_0(x))}{h_n} \\
&\leq \sup_{g \in \mathcal{G}_\rho} \frac{|v_n(x)| g(F_0(x))}{F_0(x)} \\
&\leq \sup_{g \in \mathcal{G}_\rho} \frac{|v_n(x)| g(\gamma F_0(x))}{\gamma F_0(x) \phi(x)} \\
&\leq \sup_{g \in \mathcal{G}_\rho} \frac{|v_n(x) \phi(x)| g(\gamma F_0(x))}{\gamma F_0(x) \phi(x)} \\
&\leq \|v_n\|_\phi \sup_{g \in \mathcal{G}_\rho} \frac{g(\gamma F_0(x))}{\gamma F_0(x) \phi(x)} \\
&\leq \left( \frac{\|v\|_\phi + c}{\gamma} \right) \frac{g_\rho(\gamma F_0(x))}{F_0(x) \phi(x)} \tag{39}
\end{aligned}$$

for some suitable constant  $c \in (0, \infty)$  being independent of  $n \in \mathbb{N}$ , and  $\gamma$  as in condition (b) of Assumption 2.3. Finally, for every  $x \in (F_0^\rightarrow(0), F_0^\leftarrow(1))$  with  $(F_0 + h_n v_n)(x) = 0$  (i.e. in particular with  $h_n = F_0(x)/|v_n(x)|$ ) we obtain by the concavity of all  $g \in \mathcal{G}_\rho$  and part (ii) of Theorem 2.2

$$\sup_{g \in \mathcal{G}_\rho} \int_{I_n(x)} \frac{\mathbb{1}_{[0, t_k] \cup [1 - t_k, 1]}(s) g'(s)}{h_n} ds \leq \sup_{g \in \mathcal{G}_\rho} \int_0^{F_0(x)} \frac{g'(s)}{h_n} ds$$

$$\begin{aligned}
&= \sup_{g \in \mathcal{G}_\rho} \frac{g(F_0(x))}{h_n} \\
&= \sup_{g \in \mathcal{G}_\rho} \frac{g(F_0(x)) |v_n(x)|}{F_0(x)} \\
&\leq \sup_{g \in \mathcal{G}_\rho} \frac{g(\gamma F_0(x)) |v_n(x)|}{\gamma F_0(x)} \\
&= \sup_{g \in \mathcal{G}_\rho} \frac{g(\gamma F_0(x)) |v_n(x) \phi(x)|}{\gamma F_0(x) \phi(x)} \\
&\leq \|v_n\|_\phi \sup_{g \in \mathcal{G}_\rho} \frac{g(\gamma F_0(x))}{\gamma F_0(x) \phi(x)} \\
&\leq \left( \frac{\|v\|_\phi + c}{\gamma} \right) \frac{g_\rho(\gamma F_0(x))}{F_0(x) \phi(x)} \tag{40}
\end{aligned}$$

for some suitable constant  $c \in (0, \infty)$  being independent of  $n \in \mathbb{N}$ , and  $\gamma$  as in condition (b) of Assumption 2.3. So we indeed have  $G_{n,k} \leq G$ . This completes the proof of Theorem 2.4.

#### 4.4. Proof of Theorem 2.7

Let  $(v_n)$  be a sequence in  $D_{\phi, F_0}$  with  $\|v_n - v\|_\phi \rightarrow 0$  for some  $v \in D_{\phi, F_0}$ , and let  $(h_n)$  be a sequence in  $(0, \infty)$  with  $h_n \rightarrow 0$  such that  $F_0 + h_n v_n \in \mathbb{F}_\mathcal{X}$  for every  $n$ . We have to show

$$\lim_{n \rightarrow \infty} \left| \int_{F_0^\rightarrow(0)}^{F_0^\leftarrow(1)} \frac{g((F_0 + h_n v_n)(x)) - g(F_0(x))}{h_n} dx - \int_{F_0^\rightarrow(0)}^{F_0^\leftarrow(1)} g'(F_0(x)) v(x) dx \right| = 0. \tag{41}$$

For  $\gamma \in (0, 1)$  as in condition (b) of Assumption 2.6 let us fix a sequence  $(t_k)$  in  $(0, \gamma \wedge \frac{1}{2})$  with  $t_k \downarrow 0$  and  $F^\leftarrow(t_k) = F^\rightarrow(t_k)$  as well as  $F^\leftarrow(1 - t_k) = F^\rightarrow(1 - t_k)$ . Using the notations  $g_k$ ,  $a_n(g)$ ,  $a_{n,k}(g)$ ,  $b(g)$  and  $b_k(g)$  as defined respectively by (26), (27), (28), (29), (30), we may adopt the line of reasoning from the proof of Theorem 2.4. That is, it suffices to show that the following analogues of (31)–(33) hold:

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} |a_n(g) - a_{n,k}(g)| = 0, \tag{42}$$

$$\lim_{n \rightarrow \infty} |a_{n,k}(g) - b_k(g)| = 0 \quad \text{for all } k \in \mathbb{N}, \tag{43}$$

$$\lim_{k \rightarrow \infty} |b_k(g) - b(g)| = 0. \tag{44}$$

Assertion (42) can be shown exactly in the same way as (31) was shown in Steps 3–4 in Section 4.3, where in the present setting  $\mathcal{G}_\rho$  reduces to the singleton  $\{g\}$ . By continuity and concavity of  $g$ , we have

$$g'(t) \leq \frac{\int_{(1-\lambda)t}^t g'(s) ds}{\lambda t} = \frac{g(t) - g((1-\lambda)t)}{\lambda t} \leq \frac{g(\lambda t)}{\lambda t} \quad \text{for all } t, \lambda \in (0, 1). \tag{45}$$

Then (44) can be derived in the same way as (33) was derived in Step 1 in Section 4.3.

Thus, for (41) it remains to show (43). For fixed  $k$ , and arbitrary  $n$ , we have

$$\begin{aligned}
&|a_{n,k}(g) - b_k(g)| \\
&\leq \int_{F_0^\rightarrow(0)}^{F_0^\leftarrow(1)} \left| \frac{g_k((F_0 + h_n v_n)(x)) - g_k(F_0(x))}{h_n} - g'(F_0(x)) v(x) \mathbb{1}_{[t_k, 1-t_k]}(F_0(x)) \right| dx.
\end{aligned} \tag{46}$$

We intend to apply the Dominated Convergence Theorem to conclude that the latter integral converges to 0 as  $n \rightarrow \infty$ . We first show that the integrand converges to zero (as  $n \rightarrow \infty$ ) Lebesgue a.e. To this end, let  $x \in (F_0^{\rightarrow}(0), F_0^{\leftarrow}(1))$ . If  $g$  is differentiable at  $F_0(x)$ , and  $F_0(x)$  belongs to  $(t_k, 1 - t_{k+1})$ , then  $(F_0 + h_n v_n)(x) \in (t_k, 1 - t_{k+1})$  for sufficient large  $n$ . This implies that the expression

$$\begin{aligned} & \frac{g_k((F_0 + h_n v_n)(x)) - g_k(F_0(x))}{h_n} \\ &= \frac{[g((F_0 + h_n v_n)(x)) - g_k(t_k)] - [g(F_0(x)) - g_k(t_k)]}{h_n} \\ &= \frac{g((F_0 + h_n v_n)(x)) - g(F_0(x))}{h_n} \end{aligned}$$

converges to  $g'(F_0(x))v(x)$  as  $n \rightarrow \infty$ . If  $F_0(x)$  does not belong to  $[t_k, 1 - t_{k+1}]$ , then  $(F_0 + h_n v_n)(x) \in \mathbb{R} \setminus [t_k, 1 - t_{k+1}]$  for sufficient large  $n$ . That means that

$$\frac{g_k((F_0 + h_n v_n)(x)) - g_k(F_0(x))}{h_n} = 0 \quad \text{for sufficient large } n.$$

Finally, the set  $F_0^{-1}(\{t_k, 1 - t_k\})$  is finite by assumption on  $t_k$ . To summarize, in view of Assumption 2.6 (a), we may find some Borel subset  $A \subset \mathbb{R}$  of Lebesgue measure zero such that

$$\lim_{n \rightarrow \infty} \frac{g_k((F_0 + h_n v_n)(x)) - g_k(F_0(x))}{h_n} = \mathbb{1}_{[t_k, 1 - t_k]}(F_0(x)) g'(F_0(x)) v(x)$$

holds for  $x \in (F_0^{\rightarrow}(0), F_0^{\leftarrow}(1)) \setminus A$ .

In the remainder of the proof, we will show that

$$G_k(x) := c_k \frac{g(\gamma F_0(x))}{F_0(x) \phi(x)}, \quad x \in \mathbb{R} \quad (47)$$

provides an integrable majorant for the integrand of the integral on the right-hand side in (46), where  $c_k \in (0, \infty)$  is some suitable constant being independent of  $n$ , and  $\gamma$  is as in condition (b) of Assumption 2.6. Then (43) will follow from an application of the Dominated Convergence Theorem. The integrability of  $G$  follows from Assumption 2.6 (b). To show that  $G$  is dominating we first use the concavity of  $g$  to get for  $x \in (F_0^{\rightarrow}(0), F_0^{\leftarrow}(1))$  that

$$\begin{aligned} \left| \frac{g_k((F_0 + h_n v_n)(x)) - g_k(F_0(x))}{h_n} \right| &= \frac{1}{h_n} \left| \int_{F_0(x)}^{F_0(x) + h_n v_n(x)} \mathbb{1}_{[t_k, 1 - t_k]}(t) g'(t) dt \right| \\ &\leq \frac{1}{h_n} \left| g'(t_k) \int_{F_0(x)}^{F_0(x) + h_n v_n(x)} \mathbb{1}_{[t_k, 1 - t_k]}(t) dt \right| \\ &\leq g'(t_k) |v_n(x)|. \end{aligned}$$

By assumption, we have  $t_k \leq \gamma$ . So we may conclude from (45) and the monotonicity of  $g$  that for  $x \in (F_0^{\rightarrow}(0), F_0^{\leftarrow}(1))$

$$g'(t_k) |v_n(x)| \leq \frac{g(F_0(x) t_k) |v_n(x)|}{F_0(x) t_k} \leq \frac{g(F_0(x) \gamma) \|v_n\|_{\phi}}{F_0(x) t_k \phi(x)}.$$

Analogously we obtain for  $x \in (F_0^{\rightarrow}(0), F_0^{\leftarrow}(1))$

$$g'(F_0(x)) v(x) \mathbb{1}_{[t_k, 1-t_k]}(F_0(x)) \leq \frac{g(F_0(x)\gamma) \|v\|_{\phi}}{F_0(x)\gamma \phi(x)}.$$

That is, for  $c_k := \sup_n \|v_n\|_{\phi}/t_k + \|v\|/\gamma$  the function  $G_k$  defined in (47) indeed provides a Lebesgue integrable majorant for the integrand of the integral on the right-hand side in (46); notice that the sequence  $(\|v_n\|_{\phi})$  is bounded by assumption. This completes the proof of Theorem 2.7.

## A. Quasi-Hadamard differentiability and quasi-Lipschitz continuity

The following definition recalls from [8] the notion of quasi-Hadamard differentiability. Let  $\mathbf{V}$ ,  $\mathbf{V}'$  and  $\mathbf{V}''$  be vector spaces, and  $\mathbf{V}_0$  be a subspace of  $\mathbf{V}$ . Let  $\|\cdot\|_{\mathbf{V}_0}$ ,  $\|\cdot\|_{\mathbf{V}'}$ , and  $\|\cdot\|_{\mathbf{V}''}$ , be norms on  $\mathbf{V}_0$ ,  $\mathbf{V}'$ , and  $\mathbf{V}''$ , respectively.

**Definition A.1** *Let  $f : \mathbf{V}_f \rightarrow \mathbf{V}'$  be a map defined on a subset  $\mathbf{V}_f$  of  $\mathbf{V}$ , and  $\mathbb{C}_0$  be a subset of  $\mathbf{V}_0$ . Then  $f$  is said to be quasi-Hadamard differentiable at  $\theta \in \mathbf{V}_f$  tangentially to  $\mathbb{C}_0\langle\mathbf{V}_0\rangle$  if there is some continuous map  $D_{\theta; \mathbb{C}_0\langle\mathbf{V}_0\rangle}^{\text{qHad}} f : \mathbb{C}_0 \rightarrow \mathbf{V}'$  such that*

$$\lim_{n \rightarrow \infty} \left\| D_{\theta; \mathbb{C}_0\langle\mathbf{V}_0\rangle}^{\text{qHad}} f(v) - \frac{f(\theta + h_n v_n) - f(\theta)}{h_n} \right\|_{\mathbf{V}'} = 0 \quad (48)$$

*holds for each triplet  $(v, (v_n), (h_n))$ , with  $v \in \mathbb{C}_0$ ,  $(v_n) \subset \mathbf{V}_0$  satisfying  $\|v_n - v\|_{\mathbf{V}_0} \rightarrow 0$  as well as  $(\theta + h_n v_n) \subset \mathbf{V}_f$ , and  $(h_n) \subset (0, \infty)$  satisfying  $h_n \rightarrow 0$ . In this case the mapping  $D_{\theta; \mathbb{C}_0\langle\mathbf{V}_0\rangle}^{\text{qHad}} f$  is called quasi-Hadamard derivative of  $f$  at  $\theta$  tangentially to  $\mathbb{C}_0\langle\mathbf{V}_0\rangle$ .*

Notice that quasi-Hadamard differentiability tangentially to  $\mathbb{C}_0\langle\mathbf{V}_0\rangle$  clearly implies quasi-Hadamard differentiability tangentially to  $\mathbb{B}_0\langle\mathbf{V}_0\rangle$  for every  $\mathbb{B}_0 \subset \mathbb{C}_0$ . In this case,  $D_{\theta; \mathbb{B}_0\langle\mathbf{V}_0\rangle}^{\text{qHad}} f = D_{\theta; \mathbb{C}_0\langle\mathbf{V}_0\rangle}^{\text{qHad}} f|_{\mathbb{B}_0}$ . Also notice that if  $\|\cdot\|_{\mathbf{V}_0}$  provides a norm on all of  $\mathbf{V}$ ,  $\mathbb{C}_0 = \mathbf{V}_0$ , and the derivative is linear, then the notion of quasi-Hadamard differentiability at  $\theta \in \mathbf{V}_f$  tangentially to  $\mathbb{C}_0\langle\mathbf{V}_0\rangle$  coincides with the classical notion of Hadamard differentiability tangentially to  $\mathbf{V}_0$  as defined in [17], and we write  $D_{\theta; \mathbf{V}_0}^{\text{Had}} f$  in place of  $D_{\theta; \mathbf{V}_0\langle\mathbf{V}_0\rangle}^{\text{qHad}} f$ . We stress the fact that in general  $D_{\theta; \mathbf{V}_0}^{\text{Had}} f$  is not the same as  $D_{\theta; \mathbf{V}_0\langle\mathbf{V}_0\rangle}^{\text{qHad}} f$ , because in the latter case the norm  $\|\cdot\|_{\mathbf{V}_0}$  may only be defined on  $\mathbf{V}_0$ . The preceding discussion shows in particular that quasi-Hadamard differentiability is a weaker notion of “differentiability” than the classical (tangential) Hadamard differentiability in the sense of [17]. However, it was shown in [8] that this notion is still strong enough to obtain a generalized version of the Functional Delta-Method as given in Theorem 3 of [17].

The following chain rule can be proven in the same way as the chain rule in [37, Theorem 20.9]; we omit the details. In Condition (b) we will *not* insist on linearity of the Hadamard derivative. Note that Hadamard differentiability with possibly nonlinear derivative has been studied before; see, for instance, [31].

**Lemma A.2** *Let  $f : \mathbf{V}_f \rightarrow \mathbf{V}'$  be a map defined on a subset  $\mathbf{V}_f$  of  $\mathbf{V}$ , and  $\mathbb{C}_0$  be a subset of  $\mathbf{V}_0$ . Let  $g : \mathbf{V}_g \rightarrow \mathbf{V}''$  be a map defined on a subset  $\mathbf{V}_g$  of  $\mathbf{V}'$  with  $f(\mathbf{V}_f) \subset \mathbf{V}_g$ . Let  $\mathbf{V}'_0$  be a subset of  $\mathbf{V}'$ , and assume that the following assertions hold:*

(a) The map  $f$  is quasi-Hadamard differentiable at  $\theta \in \mathbf{V}_f$  tangentially to  $\mathbb{C}_0\langle \mathbf{V}_0 \rangle$  with quasi-Hadamard derivative  $D_{\theta; \mathbb{C}_0\langle \mathbf{V}_0 \rangle}^{\text{qHad}} f$  satisfying  $D_{\theta; \mathbb{C}_0\langle \mathbf{V}_0 \rangle}^{\text{qHad}} f(\mathbb{C}_0) \subset \mathbf{V}'_0$ .

(b) The map  $g$  is Hadamard differentiable at  $f(\theta)$  tangentially to  $\mathbf{V}'_0$  with Hadamard derivative  $D_{f(\theta); \mathbf{V}'_0}^{\text{Had}} g$ .

Then  $g \circ f : \mathbf{V}_f \rightarrow \mathbf{V}''$  is quasi-Hadamard differentiable at  $\theta$  tangentially to  $\mathbb{C}_0\langle \mathbf{V}_0 \rangle$  with quasi-Hadamard derivative  $D_{\theta; \mathbb{C}_0\langle \mathbf{V}_0 \rangle}^{\text{qHad}} g \circ f = D_{f(\theta); \mathbf{V}'_0}^{\text{Had}} g \circ D_{\theta; \mathbb{C}_0\langle \mathbf{V}_0 \rangle}^{\text{qHad}} f$ .

**Definition A.3** Let  $f : \mathbf{V}_f \rightarrow \mathbf{V}'$  be a map defined on a subset  $\mathbf{V}_f$  of  $\mathbf{V}$ . The map  $f$  is said to be quasi-Lipschitz continuous at  $\theta \in \mathbf{V}_f$  along  $\mathbf{V}_0$  if

$$\|f(\theta + u_n) - f(\theta)\|_{\mathbf{V}'} = \mathcal{O}(\|u_n\|_{\mathbf{V}_0}) \quad (49)$$

holds for every sequences  $(u_n) \subset \mathbf{V}_0 \setminus \{0_{\mathbf{V}}\}$  with  $(\theta + u_n) \subset \mathbf{V}_f$  and  $\|u_n\|_{\mathbf{V}_0} \rightarrow 0$ .

**Lemma A.4** Let  $f : \mathbf{V}_f \rightarrow \mathbf{V}'$  be a map defined on a subset  $\mathbf{V}_f$  of  $\mathbf{V}$ . Then the map  $f$  is quasi-Lipschitz continuous at  $\theta \in \mathbf{V}_f$  along  $\mathbf{V}_0$  if and only if

$$\|f(\theta + h_n v_n) - f(\theta)\|_{\mathbf{V}'} = o(h_n) \quad (50)$$

holds for every sequences  $(v_n) \subset \mathbf{V}_0$  and  $(h_n) \subset (0, \infty)$  with  $(\theta + h_n v_n) \subset \mathbf{V}_f$ ,  $\|v_n\|_{\mathbf{V}_0} \rightarrow 0$  and  $h_n \rightarrow 0$ .

**Proof** The sufficiency of (49) for (50) is obvious. In order to prove the necessity, let  $f$  satisfy (50) and suppose that  $f$  does *not* satisfy (49). Then there would exist a sequence  $(u_n) \subset \mathbf{V}_0 \setminus \{0_{\mathbf{V}}\}$  with  $(\theta + u_n) \subset \mathbf{V}_f$  and  $\|u_n\|_{\mathbf{V}_0} \rightarrow 0$ , and a subsequence  $(u_{n_k}) \subset (u_n)$  such that

$$z_k := \|f(\theta + u_{n_k}) - f(\theta)\|_{\mathbf{V}'} / \|u_{n_k}\|_{\mathbf{V}_0} \rightarrow \infty, \quad k \rightarrow \infty.$$

To verify a contradiction, we set  $h_k := \|u_{n_k}\|_{\mathbf{V}_0} z_k$  and  $v_k := u_{n_k} / (\|u_{n_k}\|_{\mathbf{V}_0} z_k)$ , where we assume without loss of generality  $\|u_{n_k}\|_{\mathbf{V}_0} \in (0, 1]$  and  $z_k > 0$  for all  $k \in \mathbb{N}$ . Then, on one hand, we have  $\theta + h_k v_k (= \theta + u_{n_k}) \in \mathbf{V}_f$  and  $\|v_k\|_{\mathbf{V}_0} \rightarrow 0$ . On the other hand, we have

$$\begin{aligned} h_k &= \|u_{n_k}\|_{\mathbf{V}_0} z_k \\ &= \|f(\theta + u_{n_k}) - f(\theta)\|_{\mathbf{V}'} \\ &\leq \|f(\theta + \|u_{n_k}\|_{\mathbf{V}_0}^{1/2} \{u_{n_k} / \|u_{n_k}\|_{\mathbf{V}_0}^{1/2}\}) - f(\theta)\|_{\mathbf{V}'} / \|u_{n_k}\|_{\mathbf{V}_0}^{1/2}. \end{aligned} \quad (51)$$

Since  $\theta + \|u_{n_k}\|_{\mathbf{V}_0}^{1/2} \{u_{n_k} / \|u_{n_k}\|_{\mathbf{V}_0}^{1/2}\} (= \theta + u_{n_k}) \in \mathbf{V}_f$ ,  $\|u_{n_k} / \|u_{n_k}\|_{\mathbf{V}_0}^{1/2}\|_{\mathbf{V}_0} \rightarrow 0$  and  $\|u_{n_k}\|_{\mathbf{V}_0}^{1/2} \rightarrow 0$ , we may conclude from (51) and (50) that  $h_k \rightarrow 0$ . Thus, in view of  $0 < h_k = \|f(\theta + u_{n_k}) - f(\theta)\|_{\mathbf{V}'}$  for every  $k \in \mathbb{N}$ , we obtain by (50)

$$1 = \lim_{k \rightarrow \infty} \|f(\theta + h_k v_k) - f(\theta)\|_{\mathbf{V}'} / h_k = 0.$$

This is a contradiction.  $\square$

It is an immediate consequence of Lemma A.4 that quasi-Lipschitz continuity of  $f$  at  $\theta$  along  $\mathbf{V}_0$  exactly coincides with quasi-Hadamard differentiability of  $f$  at  $\theta$  tangentially to  $\{0_{\mathbf{V}}\} \langle \mathbf{V}_0 \rangle$  with quasi-Hadamard derivative  $D_{\theta; \{0_{\mathbf{V}}\} \langle \mathbf{V}_0 \rangle}^{\text{qHad}} f(0_{\mathbf{V}}) = 0_{\mathbf{V}'}$ , where  $0_{\mathbf{V}}$  and  $0_{\mathbf{V}'}$  denote the nulls in  $\mathbf{V}$  and  $\mathbf{V}'$ , respectively. So we immediately obtain the following lemma.

**Lemma A.5** Let  $f : \mathbf{V}_f \rightarrow \mathbf{V}'$  be a map defined on a subset  $\mathbf{V}_f$  of  $\mathbf{V}$ , and  $\mathbb{C}_0$  be a subset of  $\mathbf{V}_0$  with  $0_{\mathbf{V}} \in \mathbb{C}_0$ . If  $f$  is quasi-Hadamard differentiable at  $\theta \in \mathbf{V}_f$  tangentially to  $\mathbb{C}_0\langle \mathbf{V}_0 \rangle$  with quasi-Hadamard derivative satisfying  $D_{\theta; \mathbb{C}_0\langle \mathbf{V}_0 \rangle}^{\text{qHad}} f(0_{\mathbf{V}}) = 0_{\mathbf{V}'}$ , then  $f$  is quasi-Lipschitz continuous at  $\theta$  along  $\mathbf{V}_0$ .

## B. Separability of the uniform metric on spaces of càdlàg functions

Let  $D[0, 1]$  denote the space of càdlàg functions on  $[0, 1]$ . It is endowed with the uniform metric  $d_\infty$ . For any sequence  $\underline{a} := (a_k) \subset [0, 1]$ , we consider the set  $D_{\underline{a}}[0, 1]$  of all  $v \in D[0, 1]$  whose discontinuity points belong to  $\{a_k : k \in \mathbb{N}\}$ . We want to show that any such set is  $d_\infty$ -separable. Firstly we shall focus on the sets  $D_{\underline{a}}$  based on finite sequences  $\underline{a}$ .

**Lemma B.1** *The space  $D_{\underline{a}}[0, 1]$  is  $d_\infty$ -separable if  $\underline{a}$  is a finite sequence in  $[0, 1]$ .*

**Proof** By assumption, there exists  $0 = a_0 < \dots < a_{r+1} = 1$  such that  $D_{\underline{a}}[0, 1]$  consists of all  $v \in D[0, 1]$  whose restriction to  $[0, 1] \setminus \{a_1 \dots a_{r+1}\}$  are continuous. Let  $C[a_i, a_{i+1}]$  denote the space of all continuous real-valued mappings on  $[a_i, a_{i+1}]$  for  $i \in \{0, \dots, r\}$ . Then the mapping

$$\mathbb{R} \times \prod_{i=0}^r C[a_i, a_{i+1}] \longrightarrow D_{\underline{a}}[0, 1], \quad (x, f_0, \dots, f_r) \longmapsto \sum_{i=0}^r f_i \mathbb{1}_{[a_i, a_{i+1}]} + x \mathbb{1}_{\{1\}}$$

is surjective and continuous w.r.t. the metrics  $d$  and  $d_\infty$ , where the metric  $d$  on  $\mathbb{R} \times \prod_{i=0}^r C[a_i, a_{i+1}]$  is defined by

$$d((x, f_0, \dots, f_r), (y, g_0, \dots, g_r)) := |x - y| \vee \max_{i \in \{0, \dots, r\}} \sup_{t \in [a_i, a_{i+1}]} |f_i(t) - g_i(t)|.$$

The proof is complete because the metric  $d$  is separable.  $\square$

**Lemma B.2** *Let  $\varepsilon > 0$  and  $v \in D[0, 1]$  be such that  $|v(x) - v(x_-)| \leq \varepsilon$  for every  $x \in [0, 1]$ . Then there exists some continuous mapping  $w : [0, 1] \rightarrow \mathbb{R}$  satisfying  $d_\infty(v, w) \leq 2\varepsilon$ .*

**Proof** Let  $\mathcal{W}_v$  denote the modulus of continuity of  $v$ , i.e. the mapping

$$\mathcal{W}_v : (0, 1] \longrightarrow \mathbb{R}, \quad \delta \longmapsto \sup_{|x-y| \leq \delta} |v(x) - v(y)|.$$

Since we have assumed that  $|v(x) - v(x_-)| \leq \varepsilon$  holds for any  $x \in [0, 1]$ , Lemma 12.1 (with  $\varepsilon/2$  in place of  $\varepsilon$ ) and (12.9) in [9] ensure that we may find some  $\delta_0 \in (0, 1)$  such that  $\mathcal{W}_v(\delta_0) \leq 2\varepsilon$ . Moreover, consider the following mapping

$$\eta : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} C \exp(-\frac{1}{1-x^2}) & : |x| < 1 \\ 0 & : |x| \geq 1 \end{cases},$$

where the positive constant  $C$  is chosen so that  $\int_{-1}^1 \eta(x) dx = 1$ . Extending  $v$  to a mapping  $\widehat{v}$  on  $\mathbb{R}$  by

$$\widehat{v}(x) := \begin{cases} v(0) & : x \in [-\delta_0, 0) \\ v(x) & : x \in [0, 1] \\ v(1) & : x \in (1, 1 + \delta_0] \\ 0 & : \text{otherwise} \end{cases},$$



we obtain  $\int_{\mathbb{R}} |\widehat{v}(x)| dx < \infty$ . Then it is already known from [13, p.630, Theorem 6] that the mapping

$$\widehat{v}^{\delta_0} : (-2 + \delta_0, 2 - \delta_0) \longrightarrow \mathbb{R}, \quad x \longmapsto \int_{-\delta_0}^{\delta_0} \frac{1}{\delta_0} \eta\left(\frac{z}{\delta_0}\right) \widehat{v}(x - z) dz$$

is infinitely differentiable. In particular, its restriction  $w := \widehat{v}^{\delta_0}|_{[0,1]}$  to  $[0, 1]$  is continuous. Moreover, for any  $x \in [0, 1]$ , we have

$$|w(x) - v(x)| = \left| \int_{-1}^1 \eta(z) \widehat{v}(x - \delta_0 z) dz - \widehat{v}(x) \right| \leq \int_{-1}^1 \eta(z) |\widehat{v}(x - \delta_0 z) - \widehat{v}(x)| dz. \quad (52)$$

On the one hand, if  $z \in [-1, 1]$  with  $x - \delta_0 z < 0$ , then  $x < \delta_0$  and

$$|\widehat{v}(x - \delta_0 z) - \widehat{v}(x)| = |v(0) - v(x)| \leq \mathcal{W}_v(\delta_0) \leq 2\varepsilon.$$

On the other hand, if  $z \in [-1, 1]$  with  $x - \delta_0 z > 1$ , then  $1 - x < \delta_0$  and

$$|\widehat{v}(x - \delta_0 z) - \widehat{v}(x)| = |v(1) - v(x)| \leq \mathcal{W}_v(\delta_0) \leq 2\varepsilon.$$

Finally, if  $z \in [-1, 1]$  with  $x - \delta_0 z \in [0, 1]$ , then  $|x - \delta_0 z - x| \leq \delta_0$  and

$$|\widehat{v}(x - \delta_0 z) - \widehat{v}(x)| = |v(x - \delta_0 z) - v(x)| \leq \mathcal{W}_v(\delta_0) \leq 2\varepsilon.$$

Hence we may conclude from (52)

$$|w(x) - v(x)| \leq 2\varepsilon \int_{-1}^1 \eta(z) dz = 2\varepsilon.$$

This completes the proof since  $x \in [0, 1]$  was arbitrarily chosen.  $\square$

**Theorem B.3** *The space  $D_{\underline{a}}[0, 1]$  is  $d_{\infty}$ -separable for every sequence  $\underline{a} = (a_k)$ .*

**Proof** Let  $\underline{a}_k := \{a_1, \dots, a_k\}$  for  $k \in \mathbb{N}$ . In view of Lemma B.1, it remains to show that for every  $\varepsilon > 0$ , and any  $v \in D_{\underline{a}}[0, 1]$ , there exist  $k_0 \in \mathbb{N}$  and some  $w \in D_{\underline{a}_{k_0}}[0, 1]$  such that  $d_{\infty}(v, w) \leq \varepsilon$ . For that purpose let us fix  $\varepsilon > 0$  as well as  $v \in D_{\underline{a}}[0, 1]$ . It is well known that the set  $\{x \in [0, 1] : |v(x) - v(x_-)| > \varepsilon/2\}$  is finite. Hence there are  $k_1, \dots, k_r \in \mathbb{N}$  and  $0 =: t_0 < t_1 < \dots < t_r \leq t_{r+1} := 1$  such that  $t_i = a_{k_i}$  for  $i = 1, \dots, r$ , and  $|v(x) - v(x_-)| \leq \varepsilon/2$  for  $x \in [0, 1) \setminus \{t_1, \dots, t_r\}$ . If  $1 = a_k$  for some  $k \in \mathbb{N}$ , we choose  $t_{k_r} = a_{k_r} = 1$ . Select any  $k_0 \in \mathbb{N}$  such that  $\{a_{k_1}, \dots, a_{k_r}\} \subset \{a_k : k \in \mathbb{N} \text{ with } k \leq k_0\}$ .

Next, let us define for  $i \in \{0, \dots, r\}$  the mapping

$$v_i : [t_i, t_{i+1}] \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} v(x) & : x < t_{i+1} \\ v(t_{i+1}-) & : x = t_{i+1}. \end{cases}$$

Obviously,  $v_i$  is a càdlàg function on  $[t_i, t_{i+1}]$  with  $|v_i(x) - v_i(x_-)| \leq \varepsilon/2$  for  $x \in [t_i, t_{i+1}]$ . Since any interval  $[a, b]$  with  $a < b$  is homeomorphic with  $[0, 1]$  via some strictly increasing mapping, we may find by Lemma B.2 some continuous mapping  $w_i : [t_i, t_{i+1}] \rightarrow \mathbb{R}$  satisfying  $|w_i(x) - v_i(x)| \leq \varepsilon$  for every  $x \in [t_i, t_{i+1}]$ . Then the function

$$w := \sum_{i=0}^r w_i \mathbb{1}_{[t_i, t_{i+1})} + \left( w_r(1) \mathbb{1}_{\{0\}}(v(1_-) - v(1)) + v(1) \mathbb{1}_{\mathbb{R} \setminus \{0\}}(v(1_-) - v(1)) \right) \mathbb{1}_{\{1\}}$$

belongs to  $D_{\underline{a}_{k_0}}[0, 1]$  fulfilling  $d_\infty(v, w) \leq \varepsilon$ . This completes the proof.  $\square$

Let  $C_{u,\phi,F_0}$  and  $C_{0,\phi,F_0}$  denote the sets of all  $v \in C_{\phi,F_0}$  satisfying  $\lim_{x \rightarrow \pm\infty} v(x)\phi(x) = c_\pm$  for any constants  $c_-, c_+ \in \mathbb{R}$  and  $\lim_{x \rightarrow \pm\infty} v(x)\phi(x) = 0$ , respectively.

**Corollary B.4** *The sets  $C_{u,\phi,F_0}$  and  $C_{0,\phi,F_0}$  are  $\|\cdot\|_\phi$ -separable and  $\mathcal{D}_{\phi,F_0}$ -measurable.*

**Proof** Let  $\underline{b} = (b_k)$  denote the (possibly finite) sequence of discontinuities of  $F_0$ , viewed as a subset of the compactification  $[-\infty, \infty]$  of  $\mathbb{R}$ . We may and do pick a strictly increasing homeomorphism  $\iota$  from  $[F_0^\rightarrow(0), F_0^\leftarrow(1)]$  onto  $[0, 1]$ . Let  $D_{\iota(\underline{b})}[0, 1]$  be the set of all càdlàg functions on  $[0, 1]$  whose discontinuity points belong to  $\{\iota(b_k) : k \in \mathbb{N}\}$ . The mapping  $\gamma : D_{\iota(\underline{b})}[0, 1] \rightarrow C_{u,\phi,F_0}$  defined by

$$x \mapsto \gamma(g)(x) := \begin{cases} \frac{g \circ \iota(x)}{\phi(x)} & : x \in [F_0^\rightarrow(0), F_0^\leftarrow(1)] \\ 0 & : \text{otherwise} \end{cases}$$

is surjective and continuous w.r.t. the uniform metric on  $D_{\iota(\underline{b})}[0, 1]$  and  $\|\cdot\|_\phi$ . Since the uniform metric on  $D_{\iota(\underline{b})}[0, 1]$  is separable by Theorem B.3, we may conclude that  $C_{u,\phi,F_0}$  is separable w.r.t.  $\|\cdot\|_\phi$ . The same arguments show that  $C_{0,\phi,F_0}$  is separable w.r.t.  $\|\cdot\|_\phi$ . Finally notice that  $C_{u,\phi,F_0}$  and  $C_{0,\phi,F_0}$  are closed subsets of  $D_{\phi,F_0}$  w.r.t.  $\|\cdot\|_\phi$ . This implies that these sets belong to  $\mathcal{D}_{\phi,F_0}$ ; cf. [38, hint for Problem 1.7.4]. This completes the proof.  $\square$

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